Optional Mathematics

Grade 10

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PREFACE

School education is the foundation for preparing the citizen who are loyal to the nation and nationality, committed to the norms and values of federal democratic republic, self-reliant and respecting the social and cultural diversity. It is also remarkable for developing a good moral character with the practical know-how of the use of ICT along with the application of scientific concept and positive thinking. It is also expected to prepare the citizens who are moral and ethical, disciplined, social and human value sensitive with the consciousness about the environmental conversation and sustainable development. Moreover, it should be helpful for developing the skills for solving the real life problems. This textbook 'Optional Mathematics, Grade 10' is aligned with the intent carried out by the National Curriculum Framework for School Education, 2076 and is developed in accordance with the new Secondary Level Optional Mathematics Curriculum (Grade 9-10), 2081.

This textbook was developed by Dr. Binod Prasad Pant, Dr. Eka Raj Pandit, Ms. Nirmala Gautam, Mr. Shakti Prasad Acharya and Mr. Jagannath Adhikari. It is translated by Mr. Shusil Khanal, Mr. Ram Hari Shrestha, Mr. Surath Khadka, Ms. Maiya Khadka and Mr. Jagannath Adhikari. The contribution made by Director General Mr. Yubaraj Paudel, Prof. Dr. Hari Prasad Upadhyaya, Mr. Gyanendra Ban, Ms. Anupama Sharma, Mr. Navin Poudel, Mr. Satya Narayan Maharjan, Dr. Binod Prasad Pant, Dr. Niroj Dahal, Mr. Saroj Bhakta Acharya, Ms. Indira Pandey, Mr. Binod Thapaliya, Mr. Mahesh Bhattarai and Mr. Binod Bhattarai is remarkable in bringing the book in this form. The language of this book is edited by Mr. Bijaya Kumar Ranabhat and Mr. Binod Raj Bhatta. The layout design of the book is done by Mr. Jayaram Kuikel. The Centre expresses its heartfelt gratitude to everyone involved in its development.

The textbook is a primary resource for classroom teaching. However, the teachers are also encouraged to explore other resources in addition to this book. Considerable efforts have been made to make the book helpful in achieving the expected competencies of the curriculum. Curriculum Development Centre always welcomes constructive feedback for further betterment of its publications.

2083 BS **Curriculum Development Centre**

Contents

Lesson 1

Function

1.1 Introduction

A special type of relation in which each member of the first set is related to only one member of the second set is called a function. The concept of function, which started informally from the time of ancient was formally developed later in seventeenth century with the development of calculus. Mainly in 1673 AD, Gottfried Wilhelm Leibniz used the word function, and in 1730 AD Leonhard Euler expressed the function in the form y = $f(x)$. Functions are used in many areas of our daily life such as mathematics, science, economics, management, engineering etc.

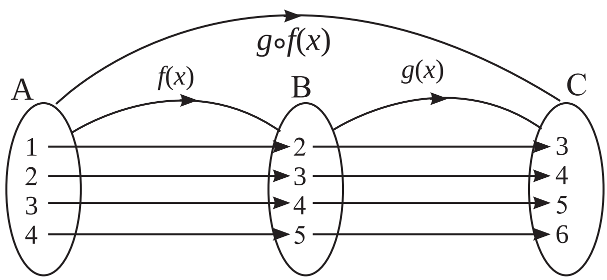
Gottfried Wilhelm Leibniz

Leonhard Euller

1.2 Composite Function

Activity 1

Observe the following mapping diagram and discuss in the groups on the basis of the given questions and draw conclusions.



a. To which members of set B is associated with each member of set A?

b. What are the images of each member of set B?

c. Is member 1 of set A associated with any element of set C?

d. Can the function from set A to set C be defined ?

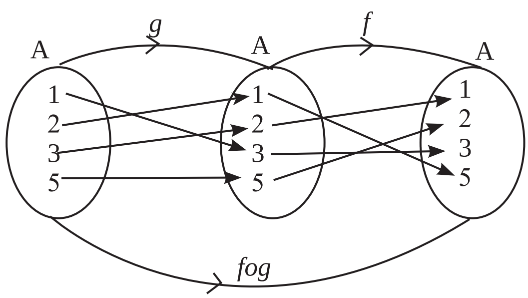
In the illustrated mapping diagram, the sets are A = {1, 2, 3, 4}, B = {2, 3, 4, 5} and C = {3, 4, 5, 6}. Here, a function $f(x)$, f: A →B = {(1, 2), (2, 3), (3, 4), (4, 5)} is defined and another function $g(x)$, g: B→C = {(2, 3), (3, 4), (4, 5), (5, 6)} is defined. Similarly, another relation A→C = {(1, 3), (2, 4), (3, 5), (4, 6)} can also be defined as a function.

Such function is called composite functions.
In this case, the composite function is denoted by $g \circ f$: A → C or $g \circ f$ or $g \circ f(x)$ or $gf(x)$ or simply gf. It is read as 'the composite function of g and f'.

If functions f: A→B and g: B→C are defined, and a relation A→C is also defined as a function, then the function A→C is called the composite function of f and g. This is denoted in the form $g \circ f$: A→C or $g \circ f(x)$ or $g \circ f$ or $gf(x)$ or gf.

Example 1

In the given mapping diagram, functions g:A→A and f: A→A are defined. Based on this, answer the following questions:



a. Write function g: A→A as a set of ordered pairs.

b. Write function f: A→A as a set of ordered pairs.

c. From the given mapping diagram, find the values of $f \circ g(1)$, $f \circ g(2)$, $f \circ g(3)$ and $f \circ g(5)$.

d. Write the composite function $f \circ g$ as a set of ordered pairs.

e. Showing the composite function $g \circ f$ in a mapping diagram, find the values of $g \circ f(1)$, $g \circ f(2)$, $g \circ f(3)$ and $g \circ f(5)$.

f. Write the composite function gof as a set of ordered pairs.

Solution: Here,

a. Function g: A→A = {(1, 3), (2, 1), (3, 2), (5, 5)}

b. Function f: A→A = {(1, 5), (2, 1), (3, 3), (5, 2)}

c. $f \circ g(1) = f(g(1)) = f(3) = 3$

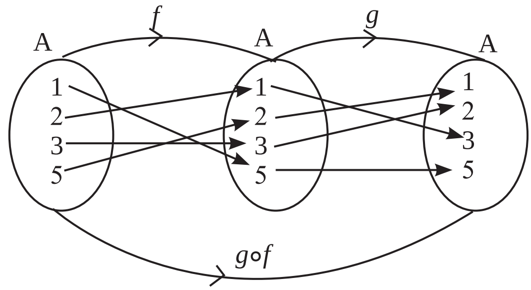
$f \circ g(2) = f(g(2)) = f(1) = 5$

$f \circ g(3) = f(g(3)) = f(2) = 1$

$f \circ g(5) = f(g(5)) = f(5) = 2$

d. Composite function $f \circ g = \{(1, 3), (2, 5), (3, 1), (5, 2)\}$

e. The mapping diagram of $g \circ f$ is as follows:



Now, $g \circ f(1) = g(f(1)) = g(5) = 5$

$g \circ f(2) = g(f(2)) = g(1) = 3$

$g \circ f(3) = g(f(3)) = g(3) = 2$

$g \circ f(5) = g(f(5)) = g(2) = 1$

f. Thus, composite function $g \circ f = \{(1, 5), (2, 3), (3, 2), (5, 1)\}$

Example 2

If $f(x) = 3x$ and $g(x) = 2x + 1$, find the value of the following composite functions.

a. $gf(x)$ **b.** $fg(x)$ **c.** $ff(x)$ **d.** $fg(1)$

Solution: Here,

$f(x) = 3x$, $g(x) = 2x + 1$

a. $gf(x) = g(f(x))$

$= g(3x)$

$= 2(3x) + 1$

$= 6x + 1$

b. $fg(x) = f(g(x))$

$= f(2x + 1)$

$= 3(2x + 1)$

$= 6x + 3$

c. $ff(x)) = f(f(x))$

$= f(3x)$

$= 3(3x)$

$= 9x$

d. $f(g(1)) = 6 \times 1 + 3$ [$\because fg(x) = 6x + 3$)]

$= 6 + 3$

$= 9$

Example 3

If $f(x) = x^2 + 2$, $g(x) = x + 14$ and $f(a) = g(a)$, find the value of a.

Solution: Here,

$f(x) = x^2 + 2$, $g(x) = x + 14$ र $f(a) = g(a)$

Now, $f(a) = g(a)$

or, $a^2 + 2 = a + 14$

or, $a^2 - a - 12 = 0$

or, $a^2 - 4a + 3a - 12 = 0$

or, $a(a - 4) + 3(a - 4) = 0$

or, $(a - 4)(a + 3) = 0$

Therefore, $a = 4$ or -3

Example 4

If $f(x) = kx + 3$, $g(x) = 2x - 4$ and $gf(2) = 34$, find the value of k.

Solution: Here,

$f(x) = kx + 3$, $g(x) = 2x - 4$ and $gf(2) = 34$

Now, $gf(x) = g(f(x)) = g(kx + 3) = 2(kx + 3) - 4 = 2kx + 6 - 4 = 2kx + 2$

Again, $g(f(2)) = 34$

or, $2k \times 2 + 2 = 34$

or, $4k + 2 = 34$

or, $4k = 32$

Therefore, $k = 8$

Example 5

If $f(x) = 2x + 5$ and $gf(x) = 18x + 41$, find the function $g(x)$.

Solution: Here,

$gf(x) = 18x + 41$

or, $g(2x + 5) = 18x + 41$

or, $g(2x + 5) = 9(2x + 5 - 5) + 41$

or, $g(2x + 5) = 9(2x + 5) - 45 + 41$

or, $g(2x + 5) = 9(2x + 5) - 4$

Therefore, $g(x) = 9x - 4$

Alternative method

$f(x) = 2x + 5$ र $gf(x) = 18x + 41$

Now, $gf(x) = 18x + 41$

or, $g(f(x)) = 18x + 41$

or, $g(2x + 5) = 18x + 41$

Suppose, $a = 2x + 5$

or, $a - 5 = 2x$

Therefore, $x = \frac{a - 5}{2}$

or, $g(a) = 18\left(\frac{a-5}{2}\right) + 41$

or, $g(a) = 9a - 45 + 41$

or, $g(a) = 9a - 4$

Therefore, $g(x) = 9x - 4$

Exercise 1.1

1. Tick (✓) the correct option for the given questions:

A. Which of the following statements is incorrect regarding the composite functions?

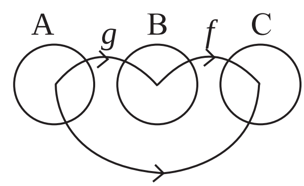
a. If functions f: A → B and g: B → C exist, and the relation A → C is also defined as a function, it is a composite function.

b. If functions g: A → B and f: B → C are one to one and onto, then, the function A → C is a composite function.

c. If functions g: A → B and f: B → C are defined, the function A → C is denoted by $g \circ f$.

d. If functions f: A → B and g: B → C are defined, the relation C → A cannot be a composite function of f and g.

B. In the given mapping diagram, how is the function defined from A to C denoted?



a. $fg(x)$ b. $gf(x)$

c. $ff(x)$ d. $gg(x)$

C. If $f(x) = \{(3, -1), (2, 0), (1, 1)\}$ and $g(x) = \{(-1, 4), (0, 5), (1, 6)\}$, what is $gf(x)$?

a. {(4, 3), (2, 5), (1, 6)} b. {(3, 4), (2, 5), (2, 6)}

c. {(3, 4), (1, 5), (2, 6)} d. {(3, 4), (2, 5), (1, 6)}

D. If $f(x) = 3x - 1$ and $g(x) = x + 1$, what is $fg(x)$?

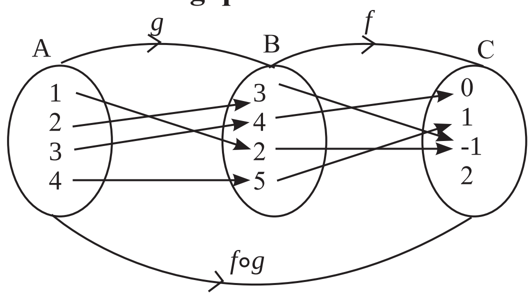
a. $3x - 1$ b. $3x + 1$ c. $3x - 2$ d. $3x + 2$

E. If $f(x) = 1 - 3x$ and $g(x) = 2x + 1$, what is $gf\left(\frac{1}{2}\right)$?

a. 3 b. 1 c. 2 d. 0

2. Define composite function with a mapping diagram.

3. In the given mapping diagram, functions g: A → B and f: B → C are defined. Answer the following questions:



a. Write function g: A → B as a set of ordered pairs.

b. Write function f: B → C as a set of ordered pairs.

c. Find the values of $f∘g(1)$, $f∘g(2)$, $f∘g(3)$, and $f∘g(4)$.

d. Write the composite function fog as a set of ordered pairs.

4. If $f(x) = \{(1, 2), (2, 3), (3, 4)\}$ and $g(x) = \{(2, 4), (3, 5), (4, 6)\}$, show $gf(x)$ in a mapping diagram and write it as a set of ordered pairs.

5. If $f(x) = 3x + 2$ and $g(x) = 5 – 3x$, find the value of the following composite functions:

a. $gf(x)$ b. $fg(x)$ c. $ff(x)$ d. $ff(3)$

6. If $f(x) = \dfrac{2x-1}{x+3}$ $(x \neq -3)$ and $g(x) = \dfrac{3x+2}{x-4}$ $(x \neq 4)$, find the value of $fg(x)$.

7. If $f(x) = 2x + 5$ and $g(x) = \dfrac{x-5}{2}$, prove that $fg(x)$ is an identity function.

8. a. If $f(x) = \dfrac{x-3}{2}$ and $fg(x) = x$, find the value of $g(x)$.

b. If $g(x) = 4 - x$ and $gh(x) = 11 - 2x$, find the value of $h(x)$.

9. a. If $f(x) = \dfrac{6}{x-2}$ $(x \neq 2)$, $g(x) = ax^2 - 1$ and $gf(5) = 7$, find the value of a.

b. If $f(x) = ax + 5$, $g(x) = 8x + 13$ and $gf(5) = 93$, find the value of a.

10. If $f(x) = kx + 1$, $g(x) = 3x - 7$ and $gf(1) = 2$, find the value of k.

11. If $f(x) = 3x - 2$, $fg(x) = 6x - 2$ and $gf(x) = 8$, find the value of x.

Answer

1. A. c	B. a	C. d	D. d	E. d				
2 - 4. Show to the teacher.		5. a. $-1-9x$	b. $17-9x$	c. $9x+8$	d. 35			
6. $\frac{5x+8}{6x-10}$	7. Show to the teacher.	8. a. $2x+3$	b. $2x-7$	9. a. 2	b. 1	10. 2	11. 2	

1.3 Inverse Function

Activity 1

Observe the given mapping diagrams, discuss the following questions and draw conclusion:

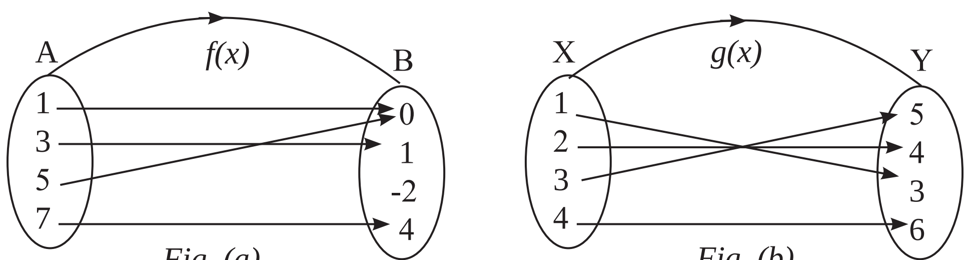


Fig. (a) *Fig. (b)*

a. Is the function $f(x)$ defined from set A to set B?

b. When the domain and codomain of set A and set B interchanged in each other, is the function defined?

c. Is the function $g(x)$ defined from set X to set Y?

d. When the domain and codomain of set X and set Y interchanged in each other, is the function defined?

In the above mapping diagram, function $f(x)$: $A \rightarrow B = \{(1, 0), (3, 1), (5, 0), (7, 4)\}$ is defined in (a), where, the domain of $f(x) = \{1, 3, 5,7\}$ and codomain of $f(x) = \{0, 1, -2, 4\}$

When the domain and codomain of function $f(x)$: $A \rightarrow B$ interchanged in each other, the relation of ordered pair and mapping diagram is formed as follows:

$R_1: B \rightarrow A = \{(0, 1), (1, 3), (0, 5), (4, 7)\}$

This relation R_1: B→A is not defined because the element 0 of the first set B is related to two elements 1 and 5. Likewise, the element −2 is not seen related with any element of the second set. Therefore, in the first function $f(x)$, the elements of the domain and the range are exchanged then the relation can not be defined as function.

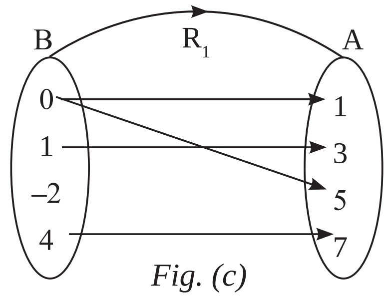


Fig. (c)

Similarly, in the mapping diagram (b), the function $g(x)$: X →Y is defined as {(1, 3), (2, 4), (3, 5), (4, 6)} where the domain of the function $g(x)$ is {1, 2, 3, 4} and the range is {3, 4, 5, 6}. In this function $g(x)$. If the elements of the domain and the range are exchanged and the order of correspondence is reversed

then, the relation can be written as:

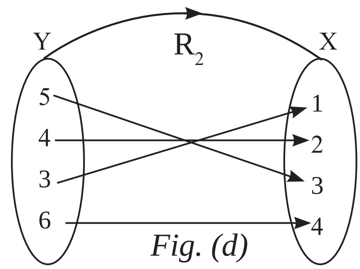


Fig. (d)

R_2: Y→X = {(3, 1), (4, 2), (5, 3), (6, 4)}

This relation R_2: Y→X is defined because each element of the first set Y is related to exactly one element of the second set. Hence, in the second function $g(x)$, even the elements of the domain and the range are exchanged the function is defined. However, the function may not always be invertible. Such an invertible function is called a one to one and onto function. The inverse relation of the function $g(x)$ is written as $g^{-1}(x)$, and g^{-1} and read as the inverse function of g.

When the domain and range of any function $f(x)$ are interchanged and the function is defined, then it is called the inverse function of $f(x)$. The inverse function of $f(x)$ is written as $f^{-1}(x)$ or f^{-1} and read as 'inverse function of f'. If the function is $y = f(x)$, then for the inverse function x and y interchange their roles. That is, if $x = f(y)$, then $y = f^{-1}(x)$.

Thought Provoking Questions: When is the inverse function $f^{-1}(x)$ of any function $f(x)$ defined? If the function has been defined as onto function, can the inverse function $f^{-1}(x)$ be defined for all functions $f(x)$? Explain. Is it necessary to a function $f(x)$ one to one onto to have its inverse function $f^{-1}(x)$?

Example 1

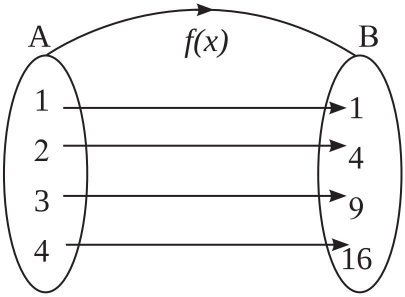
If $f(x)$ is a into function and $f(x)$ = {(1, 1), (2, 4), (3, 9), (4, 16)} then find the value of $f^{-1}(x)$ in ordered form and show it in the mapping diagram.

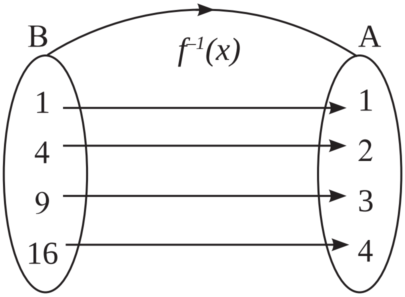
Solution: Here,

$f(x) = \{(1, 1), (2, 4), (3, 9), (4, 16)\}$

$f^{-1}(x) = \{(1, 1), (4, 2), (9, 3), (16, 4)\}$

Now, showing $f(x)$ and $f^{-1}(x)$ in mapping diagram,





Example 2

If $f(x) = 3x - 5$ is a one to one onto function where $x \in$ R and R is the set of real numbers. Is function $f(x)$ one to one and onto function? If so, find $f^{-1}(x)$.

Solution: Here,

$f(x) = 3x - 5$ is a one-one onto function where $x \in$ R and R is the set of real numbers.

Thus, each image has a different pre-image. That is, if two images are equal, their pre-images are also equal. Therefore, the function $f(x)$ is a one to one and onto function, hence the inverse function of $f(x)$ exists.

Now, $f(x) = 3x - 5$

or, $y = 3x - 5$

Again, interchanging the role of x and y, we get,

or, $x = 3y - 5$

or, $x + 5 = 3y$

or, $y = \frac{x+5}{3}$

Therefore, $f^{-1}(x) = \frac{x+5}{3}$

Example 3

If $f(x) = \frac{x+3}{2}$ is one to one and onto function and $f^{-1}(x) = x$, find the value of x.

Solution: Here,

$f(x) = \frac{x+3}{2}$

or, $y = \frac{x+3}{2}$

Again, interchanging the role of x and y, we get,

or, $x = \frac{y+3}{2}$

or, $2x = y + 3$

or, $y = 2x - 3$

Thus, $f^{-1}(x) = 2x - 3$

Now, $f^{-1}(x) = x$

or, $2x - 3 = x$

or, $2x - x = 3$

Thus, $x = 3$

Example 4

If $f(x) = \frac{3x+4}{5}$ and $g(x) = 8x - 3$ are one to one and onto function, find the value of $f^{-1}g(1)$.

Solution: Here,

$f(x) = \frac{3x+4}{5}$ and $g(x) = 8x - 3$

Now, $f(x) = \frac{3x+4}{5}$

or, $y = \frac{3x+4}{5}$

Again, interchanging the role of x and y,

or, $x = \frac{3y+4}{5}$

or, $5x = 3y + 4$

or, $5x - 4 = 3y$

or, $y = \frac{5x-4}{3}$

Thus, $f^{-1}(x) = \frac{5x-4}{3}$

Now, $f^{-1}g(x) = f^{-1}(g(x)) = f^{-1}(8x - 3)$

$= \frac{5(8x-3)-4}{3} = \frac{40x-15-4}{3}$

$= \frac{40x-19}{3}$

Again, $f^{-1}g(1) = \frac{40\times1-19}{3} = \frac{21}{3} = 7$

Thus, $f^{-1}g(1) = 7$

Example 5

If $f(x) = 2x - 3$ and $g(x) = \frac{2x-7}{3}$ are one to one and onto function. If $ff(x) = g^{-1}(x)$, find the value of x.

Solution: Here, $f(x) = 2x - 3$ and $g(x) = \frac{2x-7}{3}$

$ff(x) = f(f(x)) = f(2x - 3) = 2(2x - 3) - 3 = 4x - 6 - 3 = 4x - 9$

For, $g^{-1}(x)$

Here, $g(x) = \frac{2x-7}{3}$

or, $y = \frac{2x-7}{3}$

Again, interchanging the role of x and y, we get,

$x = \frac{2y-7}{3}$

or, $3x = 2y - 7$

or, $3x + 7 = 2y$

or, $y = \frac{3x+7}{2}$

Thus, $g^{-1}(x) = \frac{3x+7}{2}$

Now, $ff(x) = g^{-1}(x)$

or, $4x - 9 = \frac{3x+7}{2}$

or, $8x - 18 = 3x + 7$

or, $8x - 3x = 7 + 18$

or, $5x = 25$

Thus, $x = 5$

Exercise 1.2

1. **Tick (✓) the correct option for the given questions:**

A. Which statement about the inverse function is correct?

a. If the function f: A→B is defined in any case, the inverse function is also defined.

b. If the function f: A→B is defined in all cases, the inverse function B → A is also defined.

c. If the function f: A → B is a one to one and onto function, the inverse function A→B is defined.

d. If the function f: A → B is a one to one and onto function, the inverse function B → A is defined.

B. If $f(x) = \{(1, 1), (2, 4), (3,9), (4, 16)\}$ is a one to one and onto function, what is $f^{-1}(x)$?

a. {(1, 1), (4, 2), (3, 9), (16, 4)} b. {(1, 1), (4, 2), (9, 3), (16, 4)}

c. {(1, 1), (4, 2), (3, 9), (4, 16)} d. {(1, 1), (2, 4), (3, 9), (16, 4)}

C. If $f(x) = 4x + 5$ is a one to one and onto function, what is the value of $f^{-1}(x)$?

a. $5 - 4x$ b. $\frac{x-4}{5}$ c. $\frac{x+5}{4}$ d. $\frac{x-5}{4}$

D. If $g(x) = \frac{1-3x}{5}$ is a one to one and onto function, what is the value of $g^{-1}(x)$?

a. $\frac{1+3x}{5}$ b. $\frac{1-5x}{5}$ c. $\frac{1-5x}{3}$ d. $\frac{1+5x}{3}$

E. If $h(x) = \frac{1+2x}{3}$ is a one to one and onto function, what is the value of $h^{-1}(-1)$?

a. -2 b. -1 c. 1 d. 2

2. Define inverse function with a mapping diagram.
3. If $f(x)$ is a one to one and onto function and $f(x) = \{(1, 2), (2, 4), (3, 6)\}$, find $f^{-1}(x)$ in the order pair form and represent it in a mapping diagram.
4. If $f(x) = 2x-3$ is a function where $x \in R$ and R is the set of real numbers. Is this function one to one and onto? If yes, find $f^{-1}(x)$.

5. Find the value of variable x in the following cases:

a. If $f(x) = 5x - 2$ and $f^{-1}(x) = 2$, find the value of x.

b. If $g(x) = \frac{3x-7}{4}$ and $g^{-1}(x) = 5$, find the value of x.

c. If $h(x) = \frac{2x-3}{3x-5}$ and $h^{-1}(x) = 2$, find the value of x.

d. If $f(x) = \frac{x}{3+2x}$ and $f^{-1}(x) = f(x)$, find the value of x.

6. a. If $f(x) = 7x - 1$ and $g(x) = \frac{2x-3}{5}$ are one to one and onto functions, find $g^{-1}f(x)$.

b. If $f(x) = 4x - 3$ and $g(x) = \frac{x+2}{5}$ are one to one and onto functions, find $f^{-1}g^{-1}(2)$

c. If $g(x) = \frac{3x-1}{x-1}, x \neq 1$ and $h(x) = \frac{2x-1}{x-3}, x \neq 3$, find $(gh)^{-1}(x)$ and $(hg)^{-1}(x)$

7. a. If $f(x) = 2x - 3$ and $g(x) = \frac{2x-7}{3}$ are one to one and onto functions and $ff(x) = g^{-1}(x)$, find the value of x.

b. If $f(x) = 2x - 5$ and $g(x) = \frac{3x+5}{2}$ are one to one and onto functions and $ff(x) = g^{-1}(x)$, find the value of x.

c. If $f(x) = \frac{x+1}{2}$ and $g(x) = \frac{x-5}{2}$ are one to one and onto functions, and $fg^{-1}(x) = 6$, find the value of x.

Answer

1. A. d	B. b	C. d	D. c	E. a

2 – 3. Show to the teacher. 4. yes, $f^{-1}(x) = \frac{x+3}{2}$

5. a. 8 b. 2 c. 1 d. 0, –1 6.a. $\frac{35x-2}{2}$ b. $\frac{11}{4}$ c. $\frac{2x}{5-x}$, $\frac{2x+1}{5}$

7. a. 5 b. 4 c. 3

Project work

Visit three families in your surroundings. From those families, collect the data of the names of the women belonging to three generations: grandmother and grandfather, mother and father, and granddaughter and grandson. On the basis of the names of these three generations, show in a mapping diagram the composite function formed from the separate sets of women and men. Conclude whether the composite function is defined or not, prepare a report and present it in the classroom.

Lesson 2 Polynomial

2.1 Introduction

An algebraic expression in which exponent of each variable is a whole number is called a polynomial. The Greek mathematician Diophantus of Alexandria of the ancient period made special contributions to the concept of polynomials. Therefore, he is called the father of polynomials. Besides Diophantus, mathematicians from the Babylonian, Egyptian, and Chinese civilizations and Islamic mathematicians Al-Karaji and Al-Samawal, also contributed to the further development of polynomials. In the sixteenth century, François Viète, and later in the seventeenth century, René Descartes, also made contributions in this field.

Diophantus of Alexandria

Since a polynomial is a function, it is used not only in mathematics but also in daily applications such as demand, supply, velocity, acceleration, cost, production and other practical fields.

Polynomials can be linear, monomial, quadratic, cubic, quartic, etc.

A polynomial of degree n can be written as a function in the following form:

$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + a_{n-3}x^{n-3} + \ldots\ldots\ldots\ldots + a_2x^2 + a_1x^1 + a_0,$

where the coefficients $a_n, a_{n-1}, a_{n-2}, \ldots, a_1, a_0$ represent real numbers, and in x^n, n represents a whole number. In a polynomial, the exponent of the variable is either zero or a positive whole number only.

2.2 Remainder Theorem

Divide the polynomial $f(x) = x^4 - 3x^2 + 5x - 4$ by the divisor $d(x) = x - 2$ using the ordinary method of division. Find quotient $q(x)$ and the remainder R. Discuss and draw conclusion.

Given the polynomial $f(x) = x^4 - 3x^2 + 5x - 4$ and the divisor $d(x) = x - 2$, divide $f(x)$ by $d(x)$ using the ordinary method of division.

$$
\begin{array}{l}
x - 2)\ x^4 - 3x^2 + 5x - 4\ (x^3 + 2x^2 + x + 7 \\
\quad x^4 - 2x^3 \\
\quad (-)\quad (+) \\
\hline
\qquad 2x^3 - 3x^2 + 5x - 4 \\
\qquad 2x^3 - 4x^2 \\
\qquad (-)\quad (+) \\
\hline
\qquad\qquad x^2 + 5x - 4 \\
\qquad\qquad x^2 - 2x \\
\qquad\qquad (-)\quad (+) \\
\hline
\qquad\qquad\qquad 7x - 4 \\
\qquad\qquad\qquad 7x - 14 \\
\qquad\qquad\qquad (-)\quad (+) \\
\hline
\qquad\qquad\qquad\quad 10
\end{array}
$$

Thus, for polynomial $f(x) = x^4 - 3x^2 + 5x - 4$ and divisor $d(x) = x - 2$, the value of quotient $q(x) = x^3 + 2x^2 + x + 7$ and remainder $R = 10$. Thus, in this problem, $x^4 - 3x^2 + 5x - 4 = (x^3 + 2x^2 + x + 7)(x - 2) + 10$. When the polynomial $f(x)$ is divided by the divisor $d(x)$, if the quotient is $q(x)$ and the remainder is R, then, the relationship among the polynomials $f(x)$, $d(x)$, $q(x)$ and R is as follows:

$$f(x) = d(x) \times q(x) + R$$

or, $\frac{f(x)}{d(x)} = \frac{d(x) \times q(x)}{d(x)} + \frac{R}{d(x)}$

or, $\frac{f(x)}{d(x)} = q(x) + \frac{R}{d(x)}$

If $R = 0$, $\frac{f(x)}{d(x)} = q(x)$, the degree of $d(x)$ is less than the degree of $q(x)$.

When the polynomial $f(x)$ is divided by the divisor $d(x)$, there exist the quotient $q(x)$ and the remainder R, then, the relationship among the polynomial $f(x)$, $d(x)$, $q(x)$ and R is as follows: $f(x) = d(x) \times q(x) + R$.

For the polynomial $f(x) = x^3 + 2x^2 + 3x - 6$ substitute the values of x respectively as −1, 0, 1, −2, 2, −3, 3 and find the corresponding values of $f(x)$.

Is the value of $f(1)$ equal to zero? When the polynomial $f(x)$ is divided by the divisor $x - 1$, does the remainder become zero?

Substitute the values of x respectively as −1, 0, 1, −2, 2, −3, 3, and what will be the corresponding values of $f(x)$?

$f(-1) = (-1)^3 + 2(-1)^2 + 3(-1) - 6 = -1 + 2 - 3 - 6 = -8$

$f(0) = (0)^3 + 2(0)^2 + 3(0) - 6 = 0 + 0 + 0 - 6 = -6$

$f(1) = (1)^3 + 2(1)^2 + 3(1) - 6 = 1 + 2 + 3 - 6 = 0$

$f(-2)$ = ... = ?

$f(2)$ = ... = ?

$f(-3)$ = ... = ?

$f(3)$ = ... = ?

Thus, when the value of $x = a$ is substituted in the polynomial $f(x)$, the value obtained for the function is the remainder obtained when the given polynomial is divided by $(x - a)$.

That is, when the polynomial $f(x)$ is divided by the linear polynomial $x - a$, the remainder is $f(a)$.

Remainder Theorem: When a polynomial $f(x)$ of degree n is divided by $x - a$, the remainder is $f(a)$ and the degree of the quotient is $n - 1$.

Proof

When the polynomial $f(x)$ is divided by the divisor $(x - a)$, there exist the quotient $q(x)$ and the remainder R, then, according to the ordinary division method, the relation among $f(x)$, $(x - a)$, and R is: $f(x) = q(x) \times (x - a) + R$ ... (i)

Substituting $x = a$ in equation (i)

or, $f(a) = q(a) \times (a - a) + R$

or, $f(a) = q(a) \times 0 + R$

or, $f(a) = 0 + R$

or, $f(a) = R$

Therefore, $R = f(a)$. Hence, proved.

When the dividend and divisor are given, the remainder can be found as follows:

Polynomial $f(x)$	Divisor $d(x)$	Remainder (R)
$f(x)$	$x - a$	$f(a)$
$f(x)$	$x + a = x - (-a)$	$f(-a)$
$f(x)$	$ax - b = a(x - \frac{b}{a})$	$f(\frac{b}{a})$
$f(x)$	$ax + b = a\{x - (-\frac{b}{a})\}$	$f(-\frac{b}{a})$

Example 1

Using the remainder theorem, find the remainder R obtained when the polynomial $f(x) = 4x^3 + 3x^2 - 2x + 1$ is divided by the divisor $d(x) = x - 2$.

Solution

Here, the polynomial $f(x) = 4x^3 + 3x^2 - 2x + 1$ and the divisor $d(x) = x - 2$.

Now, since the divisor $d(x) = x - 2$ comparing $x - 2$ with $x - a$, we get $a = 2$.

Again, according to the remainder theorem,

Remainder R $= f(a) = f(2) = 4(2)^3 + 3(2)^2 - 2(2) + 1$

$= 4 \times 8 + 3 \times 4 - 4 + 1$

$= 32 + 12 - 4 + 1$

$= 32 + 12 - 4 + 1$

Thus, remainder R = 41

Example 2

Using the remainder theorem, find the remainder R obtained when the polynomial $f(x) = 2x^4 - 7x^3 + 12x^2 - 17x + 7$ is divided by the divisor $d(x) = 2x - 3$.

Solution

Here, the polynomial $f(x) = 2x^4 - 7x^3 + 12x^2 - 17x + 7$ and the divisor $d(x) = 2x - 3$.

Now, since the divisor $d(x) = 2x - 3 = 2(x - \frac{3}{2})$ comparing $x - \frac{3}{2}$ with $x - a$ we get $a = \frac{3}{2}$

Again, according to the remainder theorem,

Remainder R $= f(a)$

$$= f(\tfrac{3}{2}) = 2(\tfrac{3}{2})^4 - 7(\tfrac{3}{2})^3 + 12(\tfrac{3}{2})^2 - 17(\tfrac{3}{2}) + 7$$

$$= 2 \times \frac{81}{16} - 7\times \frac{27}{8} + 12 \times \frac{9}{4} - 17 \times \frac{3}{2} + 7$$

$$= \frac{81 - 189 + 216 - 204 + 56}{8} = \frac{-40}{8}$$

Thus, remainder R $= -5$

Example 3

When the polynomial $f(x) = 4x^3 - x^3 - 3x^2 + 3x - k$ is divided by the divisor $d(x) = x - 2$ remainder become 12, find the value of k.

Solution: Here,

The polynomial $f(x) = 4x^3 - 4x^2 + 3x - k$, and the divisor $d(x) = x - 2$ and remainder R = 12.

According to the remainder theorem,

Remainder R $= f(a)$

or, $R = f(2)$

or, $12 = 4(2)^3 - 3(2)^2 + 3(2) - k$

or, $12 = 4 \times 8 - 3 \times 4 + 3 \times 2 - k$

or, $12 = 32 - 12 + 6 - k$

or, $k = 32 - 12 + 6 - 12$

Thus, $k = 14$

Example 4

When the polynomial $f(x) = 2x^2 - 5x + k$ and $g(x) = x^3 - x^2 + kx + 5$ are divided by the divisor $x - 2$, remainder becomes equal, find the value of k.

Solution: Here,

Polynomials $f(x) = 2x^2\ 5x + k$ and $g(x) = x^3 - x^2 + kx + 5$ and divisor $x - 2$ and remainder is equal.

Now, $f(2) = 2 \times 2^2 - 5 \times 2 + k$

$= 2 \times 4 - 10 + k$

$= 8 - 10 + k$

$= -2 + k$

Again, $g(2) = (2)^3 - (2)^2 + k(2) + 5$

$= 8 - 4 + 2k + 5$

$= 9 + 2k$

By question, $f(2) = g(2)$

or, $-2 + k = 9 + 2k$

or, $-2 - 9 = 2k - k$

or, $-11 = k$

Therefore, $k = -11$

Exercise 2.1

1. **Tick (✓) the correct option for the given questions:**

A. If the polynomial $f(x) = x^3 + x^2 + x + 1$ and $g(x) = x + 1$, what will be the degree of $f(x) \div g(x)$?

a. 1 b. 2 c. 3 d. 4

B. When the polynomial $f(x)$ is divided by the divisor $d(x)$, the quotient is $q(x)$ and the remainder is R. Which of the following relation between the polynomials $f(x)$, $d(x)$, $q(x)$ and $r(x)$ is correct?

a. $f(x) = d(x) \times q(x) + R$ b. $f(x) = q(x) \times q(x) + R$

c. $f(x) = R \times q(x) + d(x)$ d. $f(x) = d(x) \times R + R$

C. What will be the remainder when the polynomial $f(x)$ is divided by $x - a$?

a. 0 b. 1 c. $f(x)$ d. $f(a)$

D. What will be the remainder when the polynomial $f(x)$ is divided by $ax + b$?

a. $f(\frac{a}{b})$ b. $f(-\frac{a}{b})$ c. $f(-\frac{b}{a})$ d. $f(\frac{b}{a})$

E. What will be the remainder when the polynomial $(x^3 - x^2 + 1)$ is divided by $(x - 1)$?

a. 0 b. 1 c. 2 d. 3

F. If the remainder is 4 when the polynomial $x^3 + 3x^2 - kx + 4$ is divided by $x - 2$, what is the value of k?

a. 10 b. 8 c. 4 d. 2

2. State and prove the remainder theorem.

3. Using the remainder theorem in the given expressions, find the remainder:

   a. $(x^3 - x^2 + 1) \div (x - 2)$

   b. $(x^3 + 5x^2 - 6x - 16) \div (x + 1)$

   c. $(6x^3 + 4x^2 + 3x + 4) \div (3x - 4)$

   d. $(8x^4 + 5x^3 - 2x^2 + 7x - 1) \div (2x + 6)$

4. a. For the polynomial $f(x) = 2x^3 + 3x^2 - kx + 4$ when divided by the divisor $x + 2$, the remainder is 16. Find the value of k.

   b. For the polynomial $f(x) = 8x^3 + 4x^2 + (k - 3)x - 7$, when divided by the divisor $2x - 1$, the remainder is -2. Find the value of k.

   c. For the polynomial $f(x) = x^3 - (p - 2)x^2 - px + 28$, when divided by $x + 3$, the remainder is 10. Find the value of p using the remainder theorem.

5. a. The polynomials $f(x) = 2x^3 - (k - 2)x^2 - 7x + 14$ and $g(x) = 3x^3 + 8x^2 - kx + 6$, give equal remainders when divided by $x + 2$. Using the Remainder Theorem, find the value of k.

   b. The polynomials $f(x) = 2x^3 + hx^2 - 10x + 8$ and $g(x) = 6x^3 - hx^2 + 6$, give equal remainders when divided by $2x - 1$. Using the Remainder Theorem, find the value of h.

   c. The polynomial $f(x) = x^3 + mx^2 - nx + 6$, leaves remainder 24 when divided by $x + 2$ and remainder 54 when divided by $x - 3$. Using the Remainder Theorem, find the values of m and n.

Answer

1. A. b	B. a	C. d	D. c	E. b	F. a
2. Show to the teacher.		3. a. 5	b. -6	c. $\frac{88}{3}$	d. 473
4. a. 8	b. 9	c. $\frac{3}{2}$	5. a. 1	b. 7	c. 4 and 5

2.3 Factor Theorem

Find the values of $f(2)$ and $f(3)$ for the polynomial $f(x) = x^2 - 5x + 6$? At what values of x does the polynomial $f(x)$ become zero? Discuss whether the polynomial $x^2 - 5x + 6$ is exactly divisible by $(x - 2)$ and $(x - 3)$.

Divide the polynomial $f(x) = x^3 + 7x^2 - 12x + 4$ by the divisor $d(x) = x - 1$ using the general method. Find the quotient $q(x)$ and the remainder R and establish the relation among $f(x)$, $q(x)$, $d(x)$ and R.

When the polynomial $f(x)$ is divided by the divisor $d(x)$, the quotient is $q(x)$ and the remainder is 0. Then, the relation among the polynomials $f(x)$, $d(x)$, and $q(x)$ is: $f(x) = d(x) \times q(x)$.
Therefore, if the polynomial $f(x)$ is exactly divisible by $x - a$, then, $x - a$ is a factor of the polynomial $f(x)$.

Factor Theorem: When a polynomial $f(x)$ is divided by $x - a$, $f(a) = 0$ then, $x - a$ is a factor of the polynomial $f(x)$.

Proof

When the polynomial $f(x)$ is divided by the divisor $(x - a)$, the quotient is $q(x)$ and the remainder is R. According to the division algorithm, the relation among $f(x)$, $(x - \text{a})$, $q(x)$ and R is:

$f(x) = q(x)(x - \text{a}) + \text{R}$............. (i)

If the polynomial $f(x)$ is exactly divisible by $x - a$, then the remainder R = 0.

Hence, $f(x) = q(x)(x - a) + 0$

or, $f(x) = q(x)(x - a)$

Therefore, $x - \text{a}$ is a factor of the polynomial $f(x)$. Hence proved.

Converse of Factor Theorem

If $x - \text{a}$ is a factor of the polynomial $f(x)$, then, the remainder $f(a) = 0$.

Proof

If $x - \text{a}$ is a factor of the polynomial $f(x)$, then,

$f(x) = q(x)(x - a)$............. (i)

Putting $x = a$ in equation (i),

$f(a) = q(a)(a - a)$

$\text{f}(a) = q(a) \times 0$

Hence, $f(a) = 0$. Thus, proved.

Example 1

Prove that $x - 1$ is a factor of the polynomial $f(x) = 2x^3 - 5x^2 + 4x - 1$.

Solution: Here,

Given that $x - 1$ is a factor of the polynomial $f(x) = 2x^3 - 5x^2 + 4x - 1$.

Now, $f(1) = 2(1)^3 - 5(1)^2 + 4(1) - 1$
$= 2 - 5 + 4 - 1 = 6 - 6$

Thus, $f(1) = 0$

Therefore, $x - 1$ is a factor of the polynomial $f(x) = 2x^3 - 5x^2 + 4x - 1$.

Example 2

If $x + 1$ is a factor of the polynomial $f(x) = x^3 - kx^2 + 3x + 6$, find the value of k.

Solution: Here,

Given that $x + 1$ is a factor of the polynomial $f(x) = f(x) = x^3 - kx^2 + 3x + 6$.

Now, $f(-1) = (-1)^3 - k(-1)^2 + 3(-1) + 6$

or, $0 = -1 - k - 3 + 6$

or, $0 = 2 - k$

Thus, $k = 2$

Example 3

If $x + 3$ is a factor of the polynomial $f(x) = 2x^k + 3x^{k-1} - 5x + 12$, find the value of k.

Solution

Given that $x + 3$ is a factor of the polynomial $f(x) = 2x^k + 3x^{k-1} - 5x + 12$

Now, $f(-3) = 2(-3)^k + 3(-3)^{k-1} - 5(-3) + 12$

or, $0 = 2(-3)^k + 3(-3)^k.(-3)^{-1} + 15 + 12$

or, $0 = (-3)^k\{2 - 1\} + 27$

or, $0 = (-3)^k + 27$

or, $(-3)^k = (-3)^3$

Thus, $k = 3$

Example 4

What number should be subtracted from the polynomial $f(x) = 2x^3 - 7x^2 + 7x - 9$ so that $x - 3$ becomes a factor?

Solution

Here, the polynomial $f(x) = 2x^3 - 7x^2 + 7x - 9$ and one factor is $x - 3$.

Let the number to be subtracted is k.

According to the question,

$f(3) = 2(3)^3 - 7(3)^2 + 7(3) - 9 - k$

or, $0 = 54 - 63 + 21 - 9 - k$

or, $0 = 3 - k$

Thus, $k = 3$

Therefore, the required number that should be subtracted is 3.

Exercise 2.2

1. **Tick (✓) the correct option from the given questions:**

   A. When a polynomial $f(x)$ is divided by a factor $(x - a)$, what is the value of $f(a)$?

   a. 0 b. 1 c. a d. $-a$

   B. When the polynomial $f(x)$ is divided by a linear factor, the remainder is $f(-k)$. What is that factor?

   a. $x - a$ b. $x + k$ c. $x - k$ d. $x + a$

   C. In which condition does the polynomial $f(x)$ have a factor $(cx - d)$?

   a. $f(\frac{c}{d}) = 0$ b. $f(-\frac{c}{d}) = 0$ c. $f(\frac{d}{c}) = 0$ d. $f(-\frac{d}{c}) = 0$

   D. If $(x - 1)$ is a factor of the polynomial, $x^3 + x^2 + p + 1$ what is the value of p?

   a. –5 b. –4 c. –3 d. –1

   E. If $(x + 3)$ is a factor of the polynomial $3x^3 + mx^2 - 7x - 3$, what is the value of m?

   a. 2 b. 3 c. 5 d. 7

2. Write the statement of the factor theorem and prove it.
3. If $x - a$ is a factor of a polynomial $f(x)$, prove that the remainder $f(a) = 0$.
4. **Using the Factor Theorem, prove the following in the given cases:**

   a. $x - 3$ is a factor of the polynomial $f(x) = 2x^2 - 11x + 15$

   b. $2x - 1$ is a factor of the polynomial $f(x) = 2x^3 - 7x^2 + 13x - 5$

   c. $2x + 1$ is a factor of the polynomial $f(x) = 2x^3 - 5x^2 + 7x + 5$

5. a. If $x + 2$ is a factor of the polynomial $f(x) = 4x^2 + kx + 8$, find the value of k.

   b. If $x + 1$ is a factor of the polynomial $f(x) = 2x^3 - kx^2 - 8x + 5$, find the value of k.

6. a. What number should be added to the polynomial $f(x) = x^3 + 6x^2 + 11x + 1$ so that $x + 1$ becomes a factor of the polynomial?

   b. What number should be added to the polynomial $f(x) = x^3 + x^2 - 17x + 7$ so that $x - 3$ becomes a factor of the polynomial?

c. What number should be subtracted from the polynomial $f(x) = x^3 + 9x^2 + 26x + 30$ so that $x + 4$ becomes a factor of the polynomial?

Answer					
1. A. a	B. b	C. c	D. c	E. d	2 – 3. Show to the teacher.
5. a. 12	b. 11	6. a. 5	b. 8	c. 6	

2.4 Rational Root Theorem

Study the given questions and discuss.

a. Does the equation $5x^3 - 6x^2 + 7x - 6 = 0$ satisfy, when the value of $x = 1$ is kept?

b. How can the factors of $5x^3 - 6x^2 + 7x - 6$ be found?

c. What are the factors of $5x^3 - 6x^2 + 7x - 6$?

A polynomial equation of degree n, $a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$ where a_n and a_0 are respectively the leading coefficient and the constant term, the possible roots of the equation are

$$\pm \frac{\text{factors of } a_0}{\text{factors of } a_n} \quad \text{i.e. } \pm \frac{p}{q}$$

This is called the rational root theorem.

Here, the possible roots of x should be taken in the lowest terms.

Using the rational root theorem, the factorization of polynomials and the solution of polynomial equations can be obtained. The values of x that satisfy the polynomial equation $f(x) = 0$ are called the roots or zeros of the equation. To factorize polynomials and solve polynomial equations, first one root should be found.

Steps for factorizing a polynomial of degree 3:

a. Arrange the terms of the polynomial $f(x)$ in descending order of the powers of x.

b. To find the first factor of the polynomial $f(x)$, find the ratio of the constant term (a_0) and the leading coefficient (a_n) i.e. $\frac{a_0}{a_n}$, and reduce it to the lowest term. After that, take the possible factors of the numerator and denominator and substitute different values of x such as $x = \pm 1, \pm 2, \pm 3, \pm\frac{1}{2}, \pm\frac{2}{3}$. Among them, for some value of x, the value of the polynomial becomes zero.

c. When the value of the polynomial becomes zero for a certain value of x, use the synthetic division method to find the quotient $q(x)$.

d. After obtaining the quotient $q(x)$ of degree 2, factorize it.

e. Finally, write the polynomial $f(x)$ as the product of all its three factors.

Example 1

Factorize: $x^3 - x^2 - 10x - 8$

Solution

Here, polynomial $f(x) = x^3 - x^2 - 10x - 8$

Now, from rational root theorem,

$$\text{Possible roots of } x = \pm\frac{\text{factors of } a_0}{\text{factors of } a_n} = \pm\frac{\text{factors of } 8}{\text{factors of } 1} = \pm\frac{1,2,4,8}{1}$$

Thus, possible rational roots are: $x = \pm1, \pm2, \pm4, \pm8$

Now, $f(-1) = x^3 - x^2 - 10x - 8 = (-1)^3 - (-1)^2 - 10(-1) - 8$

$= -1 - 1 + 10 - 8 = 0$

Thus, $(x + 1) = (x - (-1))$ is a factor of polynomial $f(x) = x^3 - x^2 - 10x - 8$

Now, from synthetic division method,

−1	1	**−1**	**−10**	**−8**
		−1	2	8
	1	−2	−8	0

Here, $q(x) = x^2 - 2x - 8$

$= x^2 - (4 - 2)x - 8$

$= x^2 - 4x + 2x - 8$

$= x(x - 4) + 2(x - 4)$

$= (x - 4)(x + 2)$

Thus, $(x + 1)$, $(x + 2)$ and $(x - 4)$ are the factors of polynomial $f(x) = x^3 - x^2 - 10x - 8$.

Example 2

Solve the equation: $(x - 1)(x^2 - 2x - 12) + 12 = 0$

Solution: Here,

Equation: $(x - 1)(x^2 - 2x - 12) + 12 = 0$

or, $x^3 - 2x^2 - 12x - x^2 + 2x + 12 + 12 = 0$

or, $x^3 - 3x^2 - 10x + 24 = 0$

Here, polynomial equation is $x^3 - 3x^2 - 10x + 24$

Now, from rational root theorem,

$$\text{Possible roots of } x = \pm\frac{\text{factors of } 24}{\text{factors of } 1} = \pm\frac{1,2,4,8,12,24}{1}$$

Thus, possible rational roots: $x = \pm1, \pm2, \pm3, \pm4, \pm6, \pm12, \pm24$

Again, $f(2) = (2)^3 - 3(2)^2 - 10(2) + 24 = 8 - 12 - 20 + 24 = 0$

Thus, $(x - 2)$ is a factor of polynomial $f(x) = x^3 - 3x^2 - 10x + 24$

Now, from synthetic division method,

2	1	**−3**	**−10**	**24**
		2	−2	−24
	1	−1	-12	0

Here, $q(x) = x^2 - x - 12$

$= x^2 - (4 - 3)x - 12$

$= x^2 - 4x + 3x - 12$

$= x(x - 4) + 3(x - 4)$

$= (x - 4)(x + 3)$

By factor theoreom $f(x) = q(x) \times d(x)$

$= (x - 4)(x + 3)(x - 2)$

Now, from the polynomial equation

$f(x) = 0$

or, $(x - 4)(x + 3)(x - 2) = 0$

or, $x - 4 = 0$, $\therefore x = 4$

or, $x + 3 = 0$, $\therefore x = -3$

or, $x - 2 = 0$, $\therefore x = 2$

Thus, $x = 2, -3, 4$

Exercie 2.3

1. Check whether 1, −1, −2, 2, −3, 3 are the roots/zeros of the polynomial $f(x) = x^3 - 6x^2 + 11x - 6$ or not. Find its factors.
2. Find the polynomial $f(x)$ whose zeros are 2, 4 and 6.
3. **Factorize:**

   a. $x^3 + 6x^2 + 11x + 6$ b. $x^3 - 3x^2 - 4x + 12$

   c. $x^3 - 8x^2 + 19x - 12$ d. $(x + 2)(x^2 - 11x + 48) - 120$

   e. $(x - 3)(3x^2 - x^2 - 10x + 2) - 10$
4. **Solve the equations:**

   a. $x^3 - 3x^2 - 10x + 24 = 0$ b. $2x^3 + 3x^2 - 11x - 6 = 0$

   c. $x^3 + 26x = 9x^2 + 24$ d. $x^3 - 4x^2 = 17x - 60$

   e. $(x + 4)(x^2 - x - 14) + 16 = 0$ f. $(x - 3)(3x^2 - 10x + 2) = 10$

Answer

1. Zeros of the polynomial $f(x)$ are 1, 2 and 3. Thus the factors of polynomial $f(x)$ have $(x - 2)$ $(x - 3)$ and $(x - 4)$.

2. $x^3 - 12x^2 - 10x - 48$ 3. a. $(x + 2)(x + 3)(x + 4)$ b. $(x + 2)(x - 2)(x - 3)$
c. $(x - 1)(x - 2)(x - 4)$ d. $(x - 2)(x - 3)(x - 4)$ e. $(x - 1)(x - 4)(3x - 4)$ 4. a. $(2, -3, 4)$
b. $(2, -3, \frac{-1}{2})$ c. $(2, 3, 4)$ d. $(3, -4, 5)$ e. $(-2, 4, -5)$ f. $(1, 4, \frac{4}{3})$

Lesson 3 Linear Programming

3.1 Introduction

Estimating maximum profit and minimum cost is a regular task. Among the various methods used for this purpose, linear programming is one method. The concept of linear programming was developed by the Russian mathematician Leonid Vitalyevich Kantorovich in 1939 AD. He used this concept during the Second World War for strategies to win the war while reducing expenses in situations of limited resources. Later, in 1975 AD, the American mathematical economist George B. Dantzig further developed this concept for the maximum utilization of resources in the American army and war operations. Because of this contribution, he is called the father of linear programming. Linear programming is used in various fields such as business, economics, engineering, transportation, industry, production, designing, and many other areas.

L.V. Kantorovich George B. Dantzig

3.2 Linear Inequalities

Activity 1

Can we write the following statement in mathematical form? If we can write it in mathematical form, can these also be represented in graphs? Discuss.

a. Suresh takes at most 2 hours to reach school from home.

b. A Nepali citizen must be at least 16 years old to obtain citizenship.

c. From Rs. 750, Rupesh wants to buy pens costing Rs. 15 each and copies costing Rs. 25 each. What is the maximum number of pens and copies he can buy?

In statement (a) above, if the time taken by Suresh to reach school from home is denoted by x, then the statement can be written in mathematical form as $x \leq 2$.

Similarly, in statement (b), if the age required for a Nepali citizen to obtain citizenship is denoted by x, then the statement can be written in mathematical form as $x \geq 16$.

Since there is only one variable in the above two cases, these inequalities are linear inequalities. Similarly, in the above statement (c), if the numbers of pens and copies are taken as x and y respectively, the given problem can be written as $15x + 25y \leq 750$. To represent linear inequalities with one and two variables on a graph, the following steps should be followed.

Steps used to represent linear inequalities in one and two variables on a graph:

Step 1: Convert the given linear inequality into an equation.

Step 2: After converting the inequality into an equation, the resulting line should be drawn on the graph. If the sign is > or <, it should be represented by a dotted line, and if the sign is ≥ or ≤, it should be represented by a solid line.

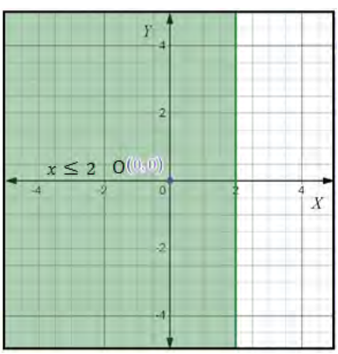
Step 3: To separate the solution region of the given inequality, take a test point that does not lie on the straight line and substitute it into the given inequality. For example: when (0, 0) is substituted in $2x + y \leq 5$, we get $0 \leq 5$ (true).

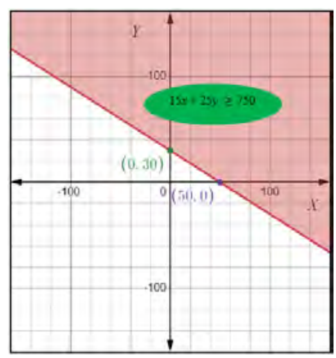
Step 4: If the mathematical statement becomes true when the test point is substituted into the given inequality, then the solution region of the inequality lies on the side of the region containing the test point. If it becomes false, then, the solution region lies on the opposite side of the region containing the test point.

Step 5: Since the true condition of the test point represents the graph of the given inequality, the solution region should be shown by shading that part on the graph.

The three linear inequalities discussed above can be represented graphically as follows:

a. Graph of $x \leq 2$ b. Graph of $x \geq 16$ c. Graph of $15x + 25y \geq 750$





Example 1

Show the following inequalities in the graph.

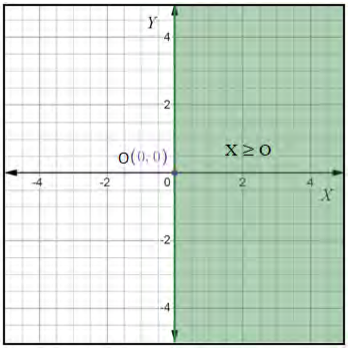
a. $x \geq 0$ b. $x \leq 0$ c. $y \geq 0$ d. $y \leq 0$

e. $x > -1$ f. $x < 2$ g. $2x + 3y \leq 6$

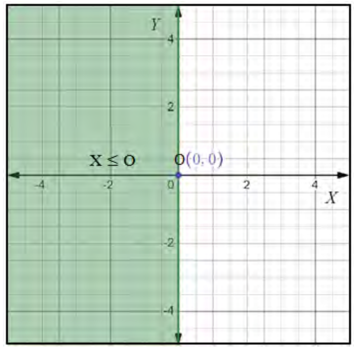
Solution

Here, the graph of a given linear inequality with one variable is shown by drawing a straight line at the point where the straight line given by the inequality intersects the X- axis or Y- axis and shading the graph towards the region given by the inequality.

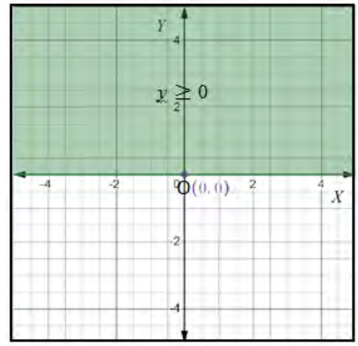
a. graph of $x \geq 0$



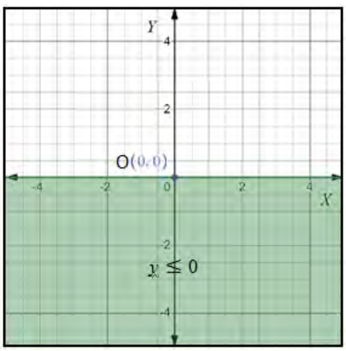
b. graph of $x \leq 0$



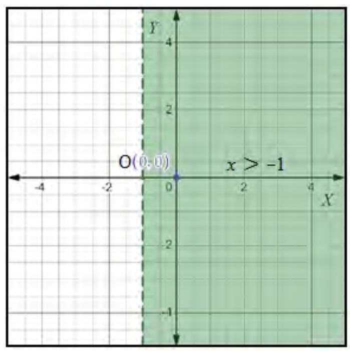
c. graph of $y \geq 0$



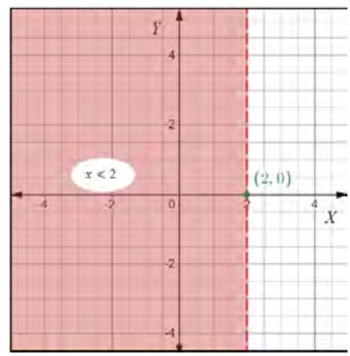
d. graph of $y \leq 0$



e. graph of $x > -1$



f. graph of $x < 2$



First, to show the given linear inequalities in the graph, we should take the equation of the boundary line related to the given inequality and find two points that lie on that line, then draw a straight line through those points. Then by taking any test point in the given inequality, the region that satisfies the inequality should be shaded in the graph.

g. $2x + 3y \leq 6$

Here, the inequality : $2x + 3y \leq 6$

The equation of the boundary line related to $2x + 3y \leq 6$ is

$$2x + 3y = 6 \qquad \text{... (i)}$$

To draw the line of equation (i) in the graph, let us find any two points that satisfy this line.

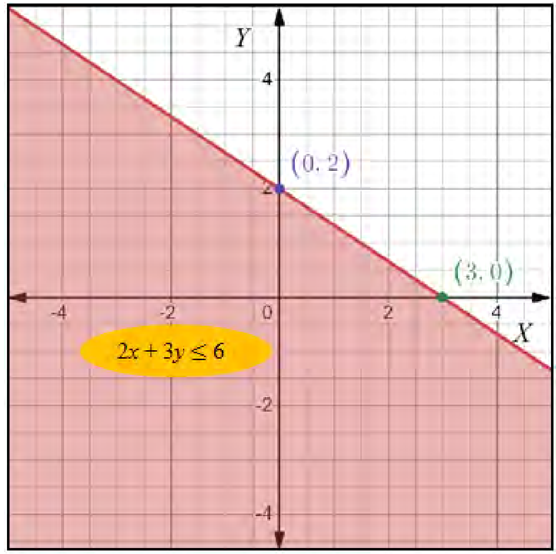
If $x = 0$ then $y = 2$

If $y = 0$ then $x = 3$

x	0	3
y	2	0

Therefore, we draw the line passing through the points (0, 2) and (3, 0) which represents the graph of $2x + 3y = 6$. Since the inequality sign is $\leq$, the line (i) is drawn as a solid line.

First, to find on which side of the line (i) the region represented by the inequality $2x + 3y \leq 6$ lies, we take a test point O(0, 0) and substitute it in the inequality $2x + 3y \leq 6$.



$2 \times 0 + 3 \times 0 \leq 6$

or, $0 \leq 6$, which is true.

Thus, when the test point O(0, 0) is substituted in the inequality $2x + 3y \leq 6$, it becomes true. Therefore, the solution region of the given inequality lies on the side of the region containing the test point O(0, 0) (here, the origin).

Hence, the solution region of the given inequality $2x + 3y \leq 6$ lies on the side of the line $2x + 3y = 6$ where the test point O(0, 0) lies. Therefore, that region is shaded, which is shown in the adjoining graph.

3.3 System of Linear Inequalities

Example 1

Plot the given inequalities in the graph. If the inequalities form a common region, what is that region called? Shade that region and discuss to draw the conclusion.

$3x + 4y \leq 12$, $2x - y \leq 2$, $x \geq 0$ and $y \geq 0$

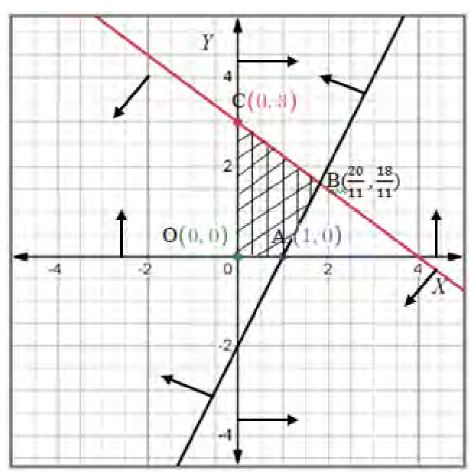
When the given inequalities $3x + 4y \leq 12$, $2x - y \leq 2$, $x \geq 0$ and $y \geq 0$ are represented in the graph, the common region of these inequalities appears as the shaded region in the following graph.

Table for first equation

x	0	4
y	3	0

Table for second equation

x	0	1
y	–2	0



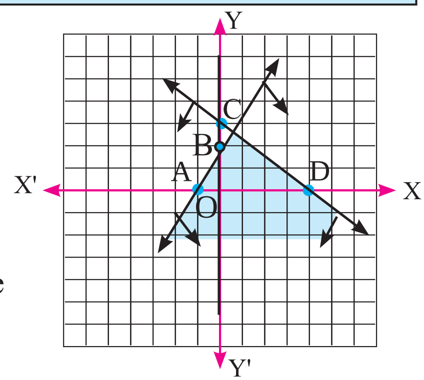
The common shaded region of all the above inequalities is the region ABCO, which is the feasible region of the inequalities.

Thus, when two or more linear inequalities are represented in the same graph, if a common solution region is obtained, such a solution region becomes a convex polygonal region. This is called a system of linear inequalities.

The vertices of the common solution region of this system of linear inequalities satisfy all the given linear inequalities. Also, from the common solution region of the system of linear inequalities, the maximum and minimum values can be determined.

When two or more linear inequalities are represented in the same graph, the common solution region in the shape of a convex polygon is called a system of linear inequalities. The vertices of the common solution region of this system of linear inequalities satisfy all the given linear inequalities.

Example 2



Find the inequalities represented by the shaded region in the given graph.

Solution

Here, from the given graph, the two points of the boundary line AB are A(−1, 0) and B(0, 2) and the two points of the dividing line CD are C(0, 3) and D(4, 0).

The equation of the line passing through AB is

$$\frac{x}{-1} + \frac{y}{2} = 1$$

$$2x - y = 2$$

Thus, the equation of the boundary line AB is $2x - y = -2$

Now, taking any point in the solution region, O(0, 0), as a test point,

The equation of the line passing through CD is

$$\frac{x}{4} + \frac{y}{3} = 1$$

$$3x + 4y = 12$$

Thus, the equation of the boundary line CD is $3x + 4y = 12$

Now, taking any point in the solution region, O(0, 0), as a test point,

$3x + 4y = 3 \times 0 + 4 \times 0 = 0 < 12$

Hence, the required inequality is $3x + 4y \leq 12$.

Therefore, the required inequalities are $2x - y \geq -2$ and $3x + 4y \leq 12$.

3.4 Solution of Linear Programming Problems

The method of linear programming is used to solve different types of business and industrial problems. It is used in industries, trade, and businesses to mobilize resources, means and capital with the objective of minimizing production cost and maximizing profit. In linear programming, efforts are made to minimize cost and maximize production and profit. In linear programming, the maximum and minimum values of a linear function are determined under given conditions called constraints. In linear programming problems, the linear function whose maximum or minimum value is to be determined is called the objective function, and the conditions expressed in the form of inequalities are called constraints.

There are two methods to solve linear programming problems. They are: graphical method and simplex method. Here, only the graphical method is used to solve linear programming problems.

To solve linear programming problems by the graphical method, the following steps are used:

Steps used to solve linear programming problems

Step 1: Plot the given linear inequalities in the same graph.

Step 2: Find the common solution region (feasible region) of all the inequalities.

Step 3: The feasible region is a convex polygon. The coordinates of the vertices of that polygon are obtained from the graph and substituted into the objective function.

Step 4: If the objective function is to be maximized, find the maximum value; if it is to be minimized, find the minimum value.

Example 3

Find the maximum and minimum values of the function: Z = 10x + 12y under the following conditions:

$x + 2y \le 12$, $3x + 2y \le 24$, $x \ge 0$, $y \ge 0$

Solution: Here,

Given function: $Z = 10x + 12y$

Inequalities: $x + 2y \le 12$, $3x + 2y \le 24$, $x \ge 0$, $y \ge 0$

The equation of boundary line related to $x + 2y \le 12$ and $3x + 2y \le 24$

$x + 2y = 12$ ... (i)

$3x + 2y = 24$ ... (ii)

To draw equation (i) in the graph, we find any two points that satisfy the line.

x	0	12
y	6	0

Thus, the line $x + 2y = 12$ passes through the points (0, 6) and (12, 0).

First, to find the region represented by the inequality $x + 2y \leq 12$, take the test point O(0, 0) and substitute it in the inequality.

$0 + 2 \times 0 \leq 12$

or, $0 \leq 12$ which is true.

Thus, when the test point O(0, 0) is substituted in the inequality $x + 2y \leq 12$, it becomes true. Therefore, the solution region of the given inequality lies on the side of the region containing the test point O(0, 0).

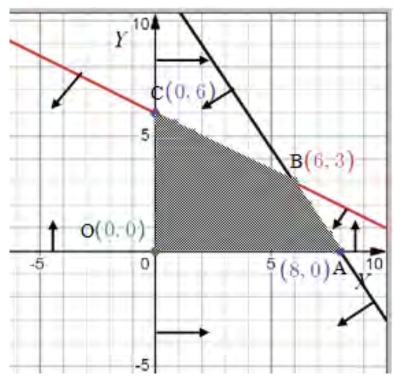
Again, to draw equation (ii) in the graph, we find any two points that satisfy the line.

x	0	8
y	12	0

Thus, the line $3x + 2y = 24$ passes through the points (0, 12) and (8, 0).

Now, to find on which side of the line (ii) the region represented by the inequality $3x + 2y \leq 24$ lies, substitute the test point (0, 0) in the inequality. $3 \times 0 + 2 \times 0 \leq 24$

or, $0 \leq 24$, which is true.



Thus, when the test point (0, 0) is substituted in the inequality $3x + 2y \leq 24$, it becomes true. Therefore, the solution region of the inequality lies on the side of the region containing the test point O(0, 0), and that region is indicated by arrows.

Similarly, the inequalities $x \geq 0$ and $y \geq 0$ represent the regions to the right of the Y- axis and above the X- axis respectively, and these regions are also indicated by arrows.

Thus, the common solution region of all the above inequalities is the shaded part shown in the adjoining graph, which is the polygon ABCO. The coordinates of the vertices of polygon ABCO are O(0, 0), A(8, 0), B(6, 3) and C(0, 6).

Now, substituting the coordinates of the vertices O(0, 0), A(8, 0), B(6, 3) and C(0, 6) into $Z = 10x + 12y$:

O(0, 0), $Z = 10 \times 0 + 12 \times 0 = 0$

A(8, 0), $Z = 10 \times 8 + 12 \times 0 = 80$

B(6, 3), $Z = 10 \times 6 + 12 \times 3 = 96$

C(0, 6), $Z = 10 \times 0 + 12 \times 6 = 72$

Therefore, the maximum value of Z is 96, which occurs at point B(6, 3), and the minimum value is 0, which occurs at point O(0, 0).

Exercise 3

1. **Tick (✓) the correct option for the given questions:**

   A. Hari earns more than Rs. 600 in a day. If Hari's one day earning is denoted by x, which one of the following inequalities is correct?

   a. $x > 600$ b. $x \geq 600$ c. $x < 600$ d. $x \leq 600$

   B. In the winter season of Kathmandu, the temperature becomes at least $-3°C$. If the winter temperature of Kathmandu is denoted by x, which one of the following inequalities represents the given statement?

   a. $x > -3°C$ b. $x \geq -3°C$ c. $x < -3°C$ d. $x \leq -3°C$

   C. Which one of the following points satisfies the inequality $2x - 3y > 5$?

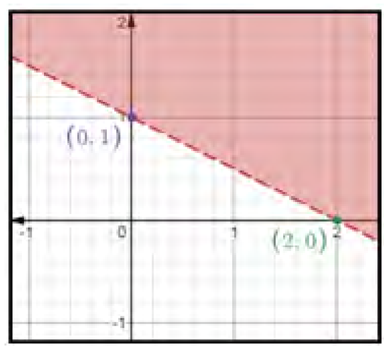
   a. (0, 0) b. (1, 1) c. (4, 3) d. (5, 1)

   D. If the objective function $Z = 3x + 4y$, among the points of the feasible region O(0, 0), A(9, 0), B(7, 3) and C(0, 8), at which point will the maximum value occur?

   a. O(0, 0) b. A(9, 0) c. B(7, 3) d. C(0, 8)

   E. Which one of the following inequalities represents the given graph?

   a. $x + y > 2$ b. $x + 2y > 2$ c. $2x + y < 2$ d. $x + 2y > 2$



2. **Draw the graphs of the following inequalities:**

   a. $x + y \geq 3$ b. $x - y \geq 0$
   c. $2x + 3y \leq 6$ d. $3x + 4y \geq 12$
   e. $x - 2y \geq 3$ f. $3x - 2y \geq 4$

3. **Draw the graphs of the following systems of inequalities and find the common solution region (feasible region):**

   a. $x + 2y \leq 6, x \geq 0$ b. $2x - y \geq 4, y \geq 0$ c. $x + y \leq 1, x - y \geq 1$

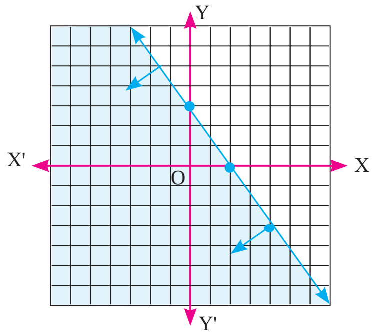
   d. $3x + 4y \leq 12, 2x - y \leq 0, x - y \leq 2,\ x \geq 0$ and $y \geq 0$

   e. $2x + 3y \leq 6, 3x - y \leq 0, x \geq 0$ and $y \geq 0$

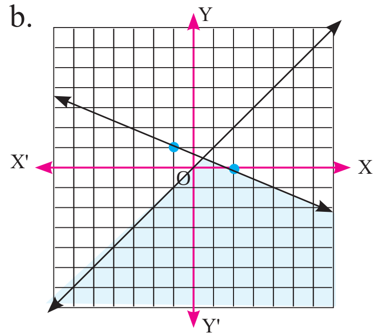
   f. $x + y \leq 2, 2x - 3y \leq 6, x \geq 0$ and $y \leq 2$

4. **Find the inequalities represented by the shaded region from the given graph:**

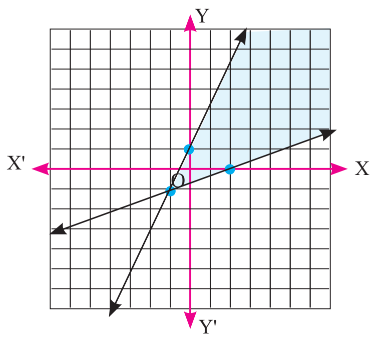
a.



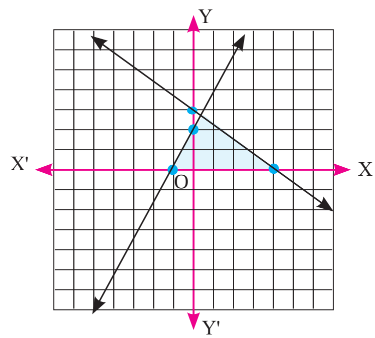
b.



c.



d.



5. **Find the maximum and minimum value of the given condition:**

a. Function: $Z = 3x + 2y$

$x + y \geq 6$

$x - y \leq 4$

$x \geq 0, y \geq 0$

b. Function: $P(x, y) = 3x + 5y$

$x + y \leq 6$

$x - y \geq 2$

$x \geq 0, y \geq 0$

c. Function: $Z = 3x + 5y$

$x - 2y \leq 1$

$x + y \leq 4$

$x \geq 0, y \geq 0$

d. Function: $F = 3x + 2y$

$x - 2y \leq 2$

$x + 2y \leq 8$

$x \geq 0, y \geq 0$

e. Function: $F(x, y) = 2x + y$

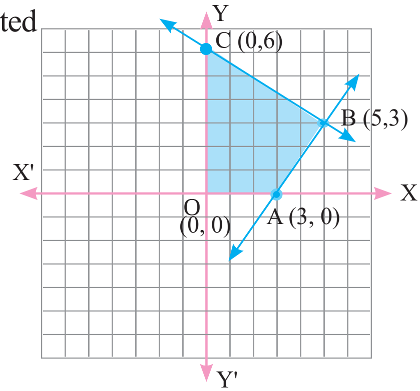
$x + 2y \leq 10$

$x + y \geq 1$

$y \leq 4$

6. Find the four inequalities which are represented by the shaded part from the given graph.

Find the maximum and minimum values of the objective function:
$F(x, y) = 20x + 30y$ from the values that satisfy those inequalities.



Answer

1. A. a B. b C. d D. c E. b

2–3. Show to the teacher.

4. a. $3x + 2y \leq 6$ b. $x + 2y \leq 0, x - y \leq 0$

c. $x - 2y \leq 1, x - 3y \leq 2$ d. $2x - y \geq -2, 3x + 4y \leq 12, y \geq 0$

5. a. Maximum Value of Z = 17 at point (5,1) and Minimum value Z = 0 at point (0, 0)

b. Maximum value of P = 22 at point (4, 2) and Minimum value P = 6 at point (2, 0)

c. Maximum value of Z = 14 at point (3, 1) and Minimum value Z = 0 at point (0, 0)

d. Maximum value of F = 18 at point $(5, \frac{3}{2})$ and Minimum value F = 0 at point (0, 0)

e. Maximum value of F = 20 at point (10, 0) and Minimum value F = –2 at point (–3, 4)

6. $3x - 2y \leq 9, 3x + 5y \leq 30, x \geq 0, y \geq 0$

Maximum value of F = 14 at point (5, 3) and Minimum value F = 0 at point (0, 0)

Lesson 4 Quadratic Equation

4.1 Introduction

An equation in which the exponent of the variable 2 is called a quadratic equation. The Persian mathematician and researcher Al-Khwarizmi (780–850 AD) of the ninth century had initial contribution on algebra. Therefore, the astronomer and mathematician Al-Khwarizmi is regarded as the father of algebra. He is considered the first mathematician to develop methods for solving quadratic equations and to present linear and quadratic equations systematically and graphically. In his book Kitab al-Jabr, Al-Khwarizmi first introduced the Hindu–Arabic numerals in the form of the word Algorithm. Later, the same book, which systematically presented linear and quadratic equations, was given the name Algebra.


Al-Khwarizmi

The equation $ax^2 + bx + c = 0$ ($a \neq 0$) is called a quadratic equation in one variable. The graph of the equation $ax^2 + bx + c = 0$ is in the shape of $\cup$ or $\cap$, which is called a parabola. The shape of the graph of a quadratic equation depends on the values of the real numbers *a*, *b*, and *c*. The point where the direction of the parabola changes is called the turning point. The turning point of the parabola is also called the vertex of the parabola.

4.2 Conversion of Quadratic Equation $y = ax^2 + bx + c$ into $y = a(x - h)^2 + k$

To convert the quadratic function $y = ax^2 + bx + c$ in the form of $y = a(x-h)^2 + k$, the following method of completing the square is used:

$$y = ax^2 + bx + c$$

or, $y = a\{x^2 + \frac{bx}{a} + \frac{c}{a}\}$

or, $y = a\{x^2 + 2.x.\frac{b}{2a} + (\frac{b}{2a})^2 - (\frac{b}{2a})^2 + \frac{c}{a}\}$

or, $y = a\{(x + \frac{b}{2a})^2 - \frac{b^2}{4a^2} + \frac{c}{a}\}$

or, $y = a\{(x + \frac{b}{2a})^2 - (\frac{b^2 - 4ac}{4a^2})\}$

or, $y = a(x + \frac{b}{2a})^2 - (\frac{b^2 - 4ac}{4a})$

or, $y = a\{(x - (-\frac{b}{2a})\}^2 + (\frac{4ac - b^2}{4a})$ ... (i)

$y = a(x - h)^2 + k$ ... (ii)

Now, comparing the equation (i) and (ii), $h = -\frac{b}{2a}$, $k = \frac{4ac - b^2}{4a}$

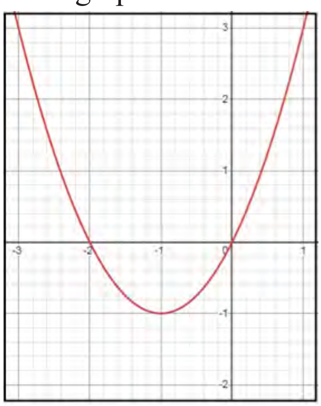
Thus, the vertices of parabola, $(h, k) = \left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$ and the axis of parabola: $h = -\frac{b}{2a}$

4.3 Sketch the Graph of Quadratic Function $y = a(x - h)^2 + k$

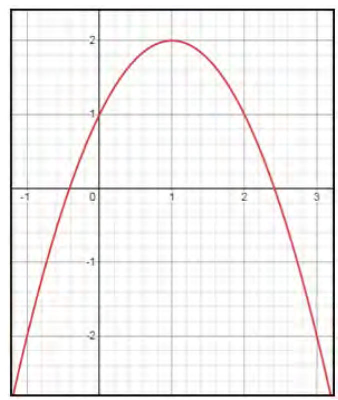
To sketch the graph of the quadratic function $y = a(x - h)^2 + k$, the following properties of the parabola are used:

1. Shape of the parabola

In the quadratic function $y = a(x - h)^2 + k$, if the value of a is positive ($a > 0$), the parabola opens upward and has a minimum point, but it does not have a maximum point. Similarly, if the value of a is negative ($a < 0$), the parabola opens downward and has a maximum point, but it does not have a minimum point. For example, in the first graph the minimum point is $(-1, -1)$ and in the second graph the maximum point is $(1, 2)$.



Graph of the quadratic function $y = -x^2 + 2x + 0$



Graph of the quadratic function $y = -x^2 + 2x + 1$

2. Vertex of the parabola

The vertex of the parabola is (h, k).

3. Intercepts of the parabola

a. **y–intercept:** In the quadratic function $y = a(x - h)^2 + k$ when $x = 0$ the y–intercept c is obtained.

b. **x–intercepts:** In the quadratic function $y = a(x - h)^2 + k$, when $y = 0$, then $y = a(x - h)^2 + k = 0$

If the roots of this quadratic equation are x_1 and x_2, then the parabola cuts the X- axis at the points $(x_1, 0)$ and $(x_2, 0)$.

4. Axis of the parabola: The axis of the parabola is $x = h$. Therefore, the parabola $y = a(x - h)^2 + k$ is symmetric about the line $x = h$.

Example 1

Quadratic function: Convert $y = x^2 - 3$ in the form of $y = a(x - h)^2 + k$ and sketch the graph.

Solution

Convert the quadratic function $y = x^2 - 3$ in the form of $y = a(x - h)^2 + k$

$y = (x - 0)^2 - 3$

Thus, for sketching the graph of quadratic function $y = (x - 0)^2 - 3$, the following properties of the parabola are used:

1. Shape of the parabola: In the quadratic function $y = (x - 0)^2 - 3$, 1 is positive $(a > 0)$, the parabola opens upwards.
2. Vertex of the parabola: The vertex of the parabola $(h, k) = (0, -3)$. The point $(0, -3)$ is the minimum point of the parabola.
3. Intercepts of the parabola:

   a. y– intercept: In the quadratic function $y = (x - 0)^2 - 3$, if $x = 0$ then, $y = (0 - 0)^2 - 3 = -3$

   Therefore, the parabola intersects the Y– axis at the point $(0, -3)$.

   b. x– intercepts: In the quadratic function $y = (x - 0)^2 - 3 = x^2 - 3$, put $y = 0$

   $x^2 - 3 = 0$

   or, $x^2 = 3$

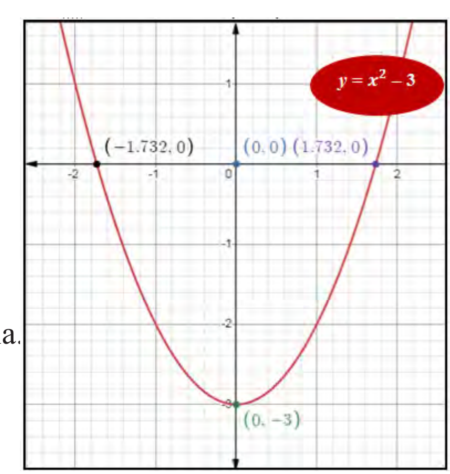
   or, $x^2 = (\pm\sqrt{3})^2$

   or, $x = \pm\sqrt{3}$

   Thus, $x = \sqrt{3} = 1.732$ or, $-\sqrt{3} = -1.732$

   Therefore, the roots of quadratic equations are (1.732 and –1.732. Hence, parabola intersects the X- axis at the points (1.732, 0) and (−1.732, 0).

4. Axis of the parabola: The axis of the parabola is the line $x = 0$ or the Y- axis, since the parabola $y = x^2 - 3$ is symmetric about the line $x = 0$.
5. Sketch of the parabola: Using the above properties, sketch the graph of the parabola.



Example 2

Quadratic function: Convert $y = x^2 - 4x + 3$ into the form $y = a(x - h)^2 + k$ and sketch the graph.

Solution

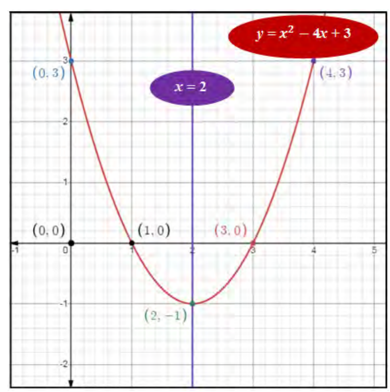
Convert the quadratic function $y = x^2 - 4x + 3$ into the form $y = a(x - h)^2 + k$

$y = x^2 - 4x + 3 = x^2 - 2.x.2 + 2^2 - 2^2 + 3 = (x - 2)^2 - 4 + 3 = (x - 2)^2 - 1$

Thus, $y = (x - 2)^2 - 1$

For sketching the graph of quadratic function $y = (x - 2)^2 - 1$, the following properties of the parabola are used:

1. **Shape of the parabola:** In the quadratic function $y = (x - 2)^2 - 1$, 1 is positive $(a > 0)$, the parabola opens upwards.
2. **Vertex of the parabola:** The vertex of the parabola $(h, k) = (2, -1)$. The point $(2, -1)$ is the minimum point of the parabola.
3. **Intercepts of the parabola:**
   a. y– intercept: In the quadratic function $y = (x - 2)^2 - 1$, if $x = 0$ then, $y = (0 - 2)^2 - 1 = (-2)^2 - 1 = 4 - 1 = 3$ Therefore, the parabola intersects the Y–axis at the point (0,3).



   b. x– intercepts: In the quadratic function $y = (x - 2)^2 - 1 = x^2 - 4x + 3$ put $y = 0$

   $x^2 - 4x + 3 = 0$

   or, $x^2 - (3 + 1)x + 3 = 0$

   or, $x^2 - 3x - 1x + 3 = 0$

   or, $x(x - 3) - 1(x - 3) = 0$

   or, $(x - 3)(x - 1) = 0$

   Thus, $x = 1$ or 3

   Therefore, the roots of quadratic equations are 1 and 3. Hence, parabola intersects the X– axis at the points (1, 0) and (3, 0).

4. Axis of the parabola: The axis of the parabola is $x = 2$, so the parabola $(x - 2)^2 - 1$ is symmetric about the line $x = 2$. Thus, the corresponding point of the point at Y axis (0, 3) is (4, 3).
5. Sketch of the parabola: Using the above properties, sketch the graph of the parabola.

Example 3

Quadratic function: Convert $y = -x^2 - 2x + 3$ in the form of $y = a(x - h)^2 + k$ and sketch the graph.

Solution

Convert the quadratic function $y = -x^2 - 2x + 3$ into the form $y = a(x-h)^2 + k$

$$y = -x^2 - 2x + 3 = -(x^2 + 2.x - 3) = -(x^2 + 2.x.1 + 1^2 - 1^2 - 3)$$
$$= -[(x+1)^2 - 1 - 3]$$
$$= -[(x+1)^2 - 4]$$

Thus, $y = -(x+1)^2 + 4$

For sketching the graph of quadratic function $y = -(x+1)^2 + 4$, the following properties of the parabola are used.

1. Shape of the parabola: In the quadratic function $y = -(x+1)^2 + 4$, coefficient of x^2 is negative, the parabola opens downwards.
2. Vertex of the parabola: The vertex of the parabola (h, k) = (–1 ,4). The point (–1, 4) is the maximum point of the parabola.
3. Intercepts of the parabola:

   a. **y- intercept:** In the quadratic function $y = -(x+1)^2 + 4$ if $x = 0$ then, $y = -(0+1)^2 + 4 = -(1)^2 + 4 = -1 + 4 = 3$

   Therefore, the parabola intersects the Y–axis at the point (0, 3).

   b. **x- intercepts:** In the quadratic function

   $y = -(x+1)^2 + 4$

   $= -x^2 - 2x + 3$ put $y = 0$

   $-x^2 - 2x + 3 = 0$

   or, $-x^2 - (3-1)x + 3 = 0$

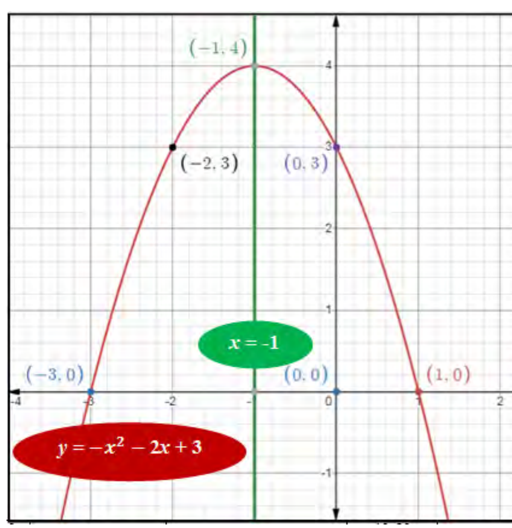
   or, $-x^2 - 3x + 1x + 3 = 0$

   or, $-x(x+3) + 1(x+3) = 0$

   or, $(-x+1)(x+3) = 0$

   Thus, $x = 1$ or -3

   Therefore, the roots of quadratic equations are 1 and –3. Hence, parabola intersects the X- axis at the points (1, 0) and (–3, 0).



4. **Axis of the parabola:** The axis of the parabola is $x = -1$, so the parabola $-(x+1)^2 + 4$ is symmetric about the line $x = -1$. Thus, the corresponding point of the point at Y axis (0, 3) is (–2, 3).
5. **Sketch of the parabola:** Using the above properties, sketch the graph of the parabola.

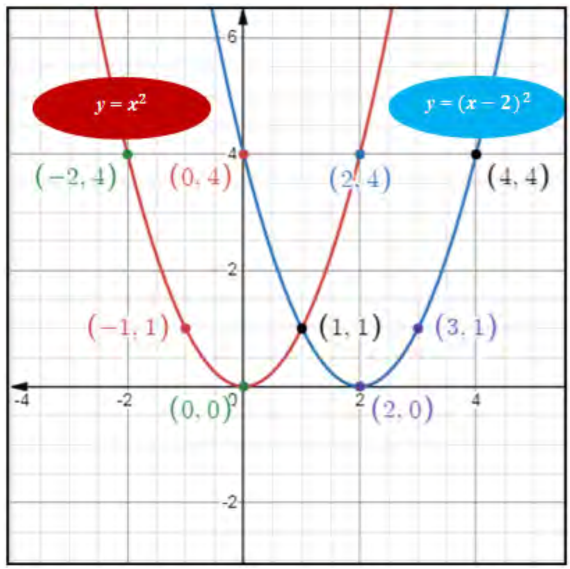
4.4 Transformation of Quadratic Function $y = x^2$

If all the points of an object are moved in the same direction, then such a transformation is called translation of that object. Generally, there are three types of translation: horizontal translation, vertical translation, and combined translation.

1. Horizontal Transformation

Activity 1

Answer the following questions based on the given graph and discuss to draw conclusion.



a. Which quadratic functions are represented by the graphs?

b. Are the graphs of the quadratic functions the same shape and size?

c. What are the coordinates of the vertices of the first and second quadratic functions?

d. How many units to the right is the vertex of the second quadratic function compared to the vertex of the first quadratic function?

e. Are the coordinates of the corresponding points of the first and second quadratic functions in the graph have the same relationship in (d)?

In the above graph, parabola is given in the first quadratic function is $y = x^2$ and the second quadratic function is $y = (x - 2)^2$.

Both parabolas have the same shape and size. The coordinates of the vertices of the first and second quadratic functions are (0, 0) and (2, 0) respectively. That is, compared to the vertex of the first quadratic function, the vertex of the second quadratic function has shifted 2 units to the right.

Similarly, the corresponding point for (–1, 1) is (1, 1) and the corresponding points for (1, 1) is (3, 1), both of which have also shifted 2 units to the right. In this way, all the remaining points have also shifted 2 units to the right.

Thus, the quadratic function $y = x^2$ becomes $y = (x - 2)^2$ after a translation of 2 units to the right.

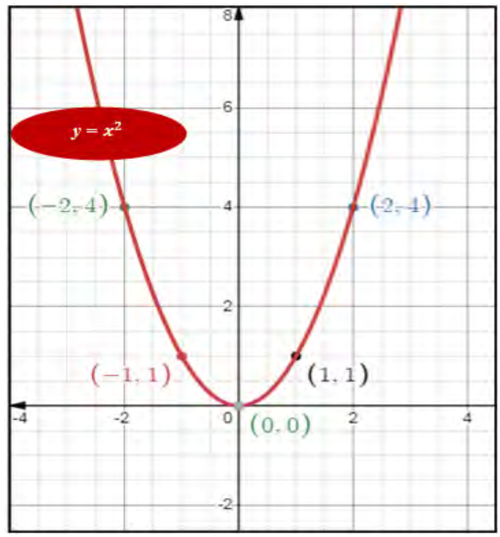
Thought Provoking Question: What will be the quadratic function and its graph look like when the quadratic function $y = x^2$ is shifted 2 units to the left? Discuss.

If the function $y = x^2$ is shifted h units to the right, the new quadratic function becomes $y = (x - h)^2$.

Similarly, if $y = x^2$ is shifted h units to the left, the new quadratic function becomes $y = (x + h)^2$. Durinig the horizontal transformation h units left and right, the parabola of the quadratic function $y = x^2$, remains the same size and the same shape.

2. Vertical Transformation

Activity 2



From the given graph, write the answer to the following questions then discuss and draw conclusion.

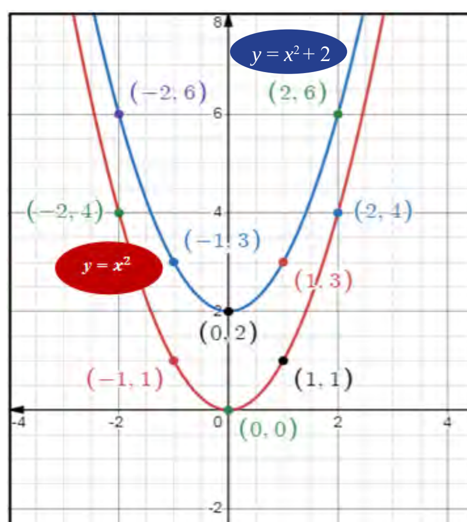
a. In the given figure, consider the quadratic function $y = x^2$ and draw it in your own graph.

b. By shifting the function $y = x^2$ vertically, create five new points by moving each point 2 units upward. Write their coordinates and show them on the same graph.

c. Prepare a curve by joining the five new points created in this way.

d. Write down the measurment and shape of the new curve thus formed.

e. Write down the new function of the shifted parabola.



In the given figure, shifting the quadratic function $y = x^2$ vertically upward by 2 units gives the new function $y = x^2$. The new five points after shifting 2 units upward are:

$(-2, 4) \rightarrow (-2, 6)$

$(-1, 1) \rightarrow (-1, 3)$

$(0, 0) \rightarrow (0, 2)$

$(1, 1) \rightarrow (1, 3)$

$(2, 4) \rightarrow (2, 6)$

Joining these points gives the parabola of $y = x^2 + 2$

By the same as, the new function of the parabola after shifting 2 units downward is $y = x^2 - 2$. If the vertices of the parabola is origin (0, 0), then the vertices of parabola after displacement is (0, ±2).

The parabola formed by displacing the quadratic function $y = x^2$ up by k units is $y = x^2 + k$ and the parabola formed by displacing it down is $y = x^2 - k$. The shape and size of the displaced parabola are the same as $y = x^2$.

3. Combined Transformation

Activity 3

Draw the graph of the quadratic function: $y = x^2$. Draw and discuss the quadratic function obtained by first translating the quadratic function $y = x^2$ by 2 units to the right and then translating it by 3 units upward and draw a conclusion.

To show the quadratic function $y = x^2$ in the graph, different values of the variable x are placed in the function $y = x^2$ and different values of the variable y are obtained. Complete the table below and draw the graph given by the quadratic function $y = x^2$.

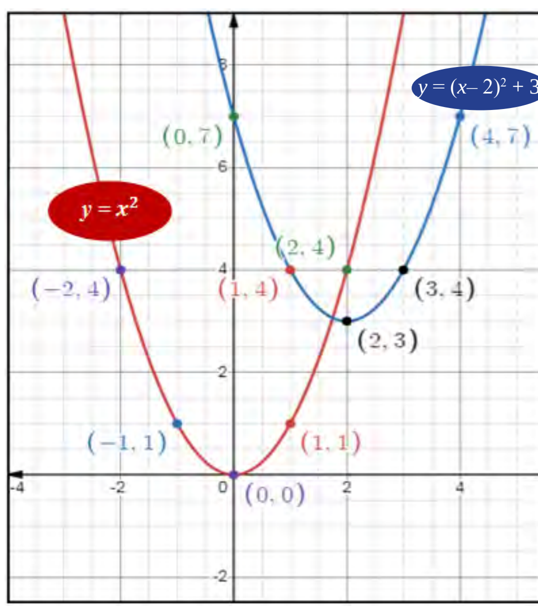
Table for the quadratic function $y = x^2$.

x	–2	–1	0	1	2
y	4	1	...	...	...

Table of the quadratic function obtained by first translating 2 units to the right and then translating 3 units upward

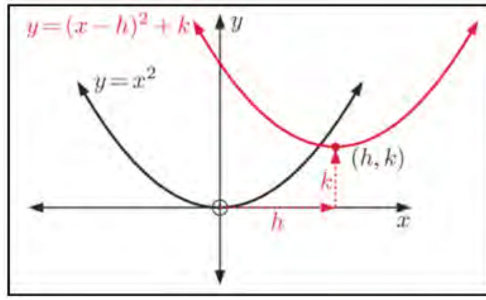
x	0	1	2	3	4
y	7	4	3	4	7

When plotting the values of x and y from both tables respectively on the graph, curves like those shown in the figure below are formed.



In the given graph, both are parabolas of the same shape. The coordinates of the vertex of the parabola $y = x^2$ is the origin O(0, 0), and the coordinates of the vertex of the parabola after the combined translation is (2, 3), and the axis of symmetry is $x = 2$. When the quadratic function $y = x^2$ is translated 2 units horizontally to the right and 3 units vertically upward, the quadratic function becomes $y = (x - 2)^2 + 3$. This is shown in the accompanying graph.

Thus, when the quadratic function $y = x^2$ is first translated h units to the right and then translated k units upward, the quadratic function becomes $y = (x - h)^2 + k$. This is the combined translation of the quadratic function $y = x^2$. The coordinates of the vertex after translation are (h, k) and the axis of symmetry is $x = h$. The quadratic function $y = (x - h)^2 + k$ is called the vertex form of a parabola.



When the quadratic function $y = x^2$ is first translated h units to the right and then translated k units upward, the quadratic function becomes $y = (x - h)^2 + k$. This is the combined translation of the quadratic function $y = x^2$. The coordinates of the vertex after translation are (h, k) and the axis of symmetry is $x = h$. The quadratic function $y = (x - h)^2 + k$ is called the vertex form of a parabola

Different condition of the quadratic function

S.N.	Equation of Parabola	Vertex	Direction of Opening of Parabola
1.	$y = x^2$	(0, 0)	Upward
2.	$y = -x^2$	(0, 0)	Downward
3.	$y = ax^2$, $a > 0$	(0, 0)	Upward
4.	$y = -ax^2$, $a > 0$	(0, 0)	Downward
5.	$y = x^2 + k$	$(0, k)$	Upward
6.	$y = -x^2 + k$	$(0, k)$	Downward
7.	$y = (x + \mathrm{k})^2$	$(-k, 0)$	Upward
8.	$y = (x - k)^2$	$(k, 0)$	Upward
9.	$y = a(x - h)^2 + k$	(h, k)	Upward

Exercise 4.1

1. Tick (✓) the correct option for the given questions:

A. What are the coordinates of the turning point of the graph of the quadratic function $y = ax^2$?

a. (0, 0) b. $(k, 0)$ c. (h, k) d. $(h, 0)$

B. What is the shape of the quadratic function $\mathrm{y} = -ax^2$ with $a > 0$?

a. ∪ b. ∩ c. ⊂ d. ⊃

C. Which one of the following line is the line symmetric of the parabola represented by $y = 2x^2$?

a. X- axis b. Y- axis c. $y = 0$ d. $y = 1$

D. What are the coordinates of the vertex of the quadratic function $y = a(x - h)^2 + k$?

a. (0, 0) b. (k, 0) c. (h, k) d. (h, 0)

E. What are the coordinates of the vertex of the parabola $y = x^2 - 2x + 1$?

a. (0, 0) b. (–1, 0) c. (1, 0) d) (1, 1)

2. Convert the following quadratic equations into the vertex form $y = (x - h)^2 + k$ and sketch the graph:

a. $y = x^2$ b. $y = 2x^2$ c. $y = -3x^2$

d. $y = x^2 + 2$ e. $y = x^2 - 5$ f. $y = (x - 3)^2$

g. $y = (x + 3)^2$ h. $y = x^2 + 2x + 1$ i. $y = x^2 - 4x + 3$

j. $y = -x^2 - 3x + 2$ k. $y = x^2 - x - 2$ l. $y = -x^2 + 7x + 12$

3. Study the given quadratic equations and answer the following questions:

a. $x^2 - 4x + 7 = 0$ b. $x^2 - 6x + 13 = 0$ c. $x^2 - 8x + 21 = 0$

d. $x^2 + 2x - 4 = 0$ e. $x^2 + 2x + 1 = 0$

i. Convert the quadratic equation into vertex form.

ii. Find the vertex of the parabola.

iii. Find the equation of the axis of symmetry.

iv. Show the graph as a translation of the quadratic function $y = x^2$.

Answer

1. A. a B. b C. b D. c E. c

2. Show to the teacher. 3. a. $(x - 2)^2 + 3$, (2, 3) and axis of symmetry $x = 2$

b. $(x - 3)^2 + 4$, (3, 4) and axis of symmetry $x = 3$ (c) $(x - 4)^2 + 5$, (4, 5) and axis of symmetry $x = 4$

c. $(x + 1)^2 - 5$, (–1, –5) and axis of symmetry $x = -1$

e. $(x + 2)^2 - 3$, (–2, –3) and axis of symmetry $x = -2$ Show the graph to the teacher.

4.5 Solution of Quadratic Equation Using Graphs

The equation $ax^2 + bx + c = 0$ is a general quadratic equation. Here, a, b, c are real numbers and $a \neq 0$. The methods for solving quadratic equations are factorization, using formula, completing the square etc. Here, we will study the method of solving the quadratic equation $ax^2 + bx + c = 0$ by the graphical method.

Example 1

Solve the equation $x^2 - 2x - 3 = 0$ by graphical method.

Solution

Given quadratic equation $x^2 - 2x - 3 = 0$

or, $x^2 = 2x + 3 = y$

$y = x^2$ ... (i)

$y = 2x + 3$ ... (ii)

Now, to show the parabola (i) and the straight line (ii) on the graph, let's put different values of x as shown in the table below and find the corresponding values of y:

From $y = x^2$

x	–3	–2	–1	0	1
y	9	4	1	0	1

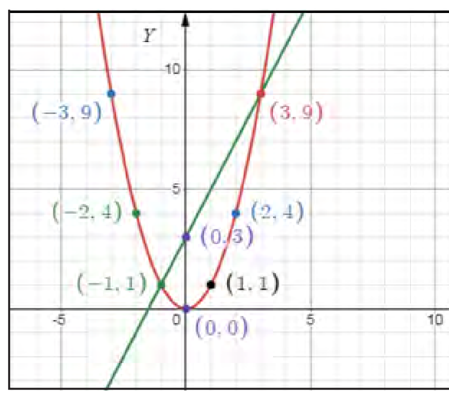
From $y = 2x + 3$

x	0	–1
y	3	1

Plot the different values of x and y from both equations above on the graph and join the points.

In the graph given alongside, the parabola $y = x^2$ intersects the line $y = 2x + 3$ at the points (−1, 1) and (3, 9). Therefore, the solutions of the quadratic equation $x^2 – 2x – 3 = 0$ are the points (–1, 0) and (3, 0).

Hence, $x = −1$ or 3 are the solutions of the quadratic equation: $x^2 – 2x – 3 = 0$.



Parabola $y = x^2$ and line $y = 2x + 3$

Another Method

Here, the given equation is $x^2 – 2x – 3 = 0$ ... (i)

Comparing equation (i) with the quadratic equation $y = ax^2 + bx + c$ we have $a = 1$, $b = –2$ and $c = –3$

Therefore, the vertex of the parabola is $V(h, k) = V\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$

$= V\left(-\frac{-2}{2\times1}, \frac{4\times1\times-3-(-2)^2}{4\times1}\right)$

$= V(1, \frac{-12 - 4}{4}) = V(1, \frac{-16}{4})$

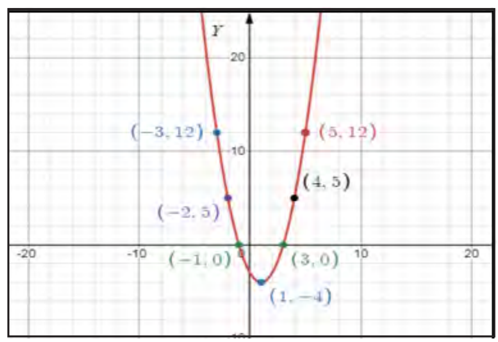
Thus, $V(h, k) = V(1, -4)$

To show the equation $x^2 - 2x - 3 = 0$ on the graph, let's put different values of x as shown in the table below and find the corresponding values of y.

x	–3	–2	–1	1	3	4	5
y	12	5	0	–4	0	5	12

Let's plot the different values of x and y on the graph and join the points.

In the given graph, the given quadratic equation $x^2 - 2x - 3 = 0$ intersects the X- axis at the points (−1, 0) and (3, 0).



Therefore, the solution points of the quadratic equation $x^2 - 2x - 3 = 0$ are the points (−1, 0) and (3, 0).

Hence, $x = -1$ or 3 are the solutions of the quadratic equation $x^2 - 2x - 3 = 0$.

4.6 Solution of Quadratic Equation and Linear Equation Using Graphs

When a quadratic equation $ax^2 + bx + c = 0$ and a linear equation $y = mx + c$ are given, we solve these two equations to find the solution. To solve these two equations graphically, we plot different values of x and y from both equations on the graph, join the points and draw the parabola and the line. Then, the points where the parabola and the line intersect each other on the graph are the solutions of the two equations.

Example 2

Solve the equations $x^2 - x - 3 = 0$ and $x + y = -2$ by graphical method.

Solution: Here,

Given equations,

$y = x^2 - x - 3$ ... (i)

$x + y = -2$ ... (ii)

Comparing equation (i) with the quadratic equation $y = ax^2 + bx + c$ we have

$a = 1$, $b = -1$ and $c = -3$.

Therefore, the vertex of the parabola is $V(h, k) = V\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$

$$= V\left(-\frac{-1}{2\times1}, \frac{4\times1\times-3-(-1)^2}{4\times1}\right)$$

$$= V\left(\frac{1}{2}, \frac{-12-1}{4}\right) = V\left(\frac{1}{2}, \frac{-13}{4}\right)$$

Thus, $V(h, k) = V\left(\frac{1}{2}, \frac{-13}{4}\right) = V(0.5, -3.25)$

To show the equation (i) and (ii) on the graph, let's put different values of x as shown in the table below and find the corresponding values of y.

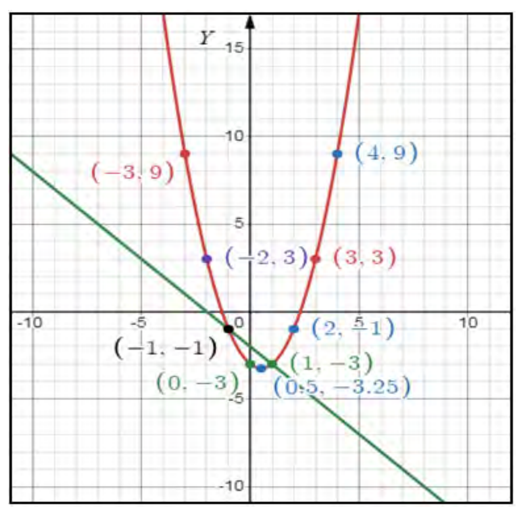
From $y = x^2 - x - 3$

x	–3	–2	–1	0.5	1	2	3	4
y	9	3	–1	–3.25	–3	–1	3	9

From $y = -2 - x$

x	–1	1
y	–1	–3

Let's plot the different values of x and y on the graph and join the points.

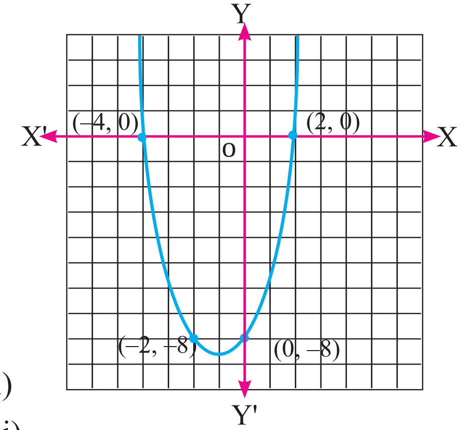


In the given graph, the line $x + y = -2$ intersects the given equation $x^2 - x - 3 = 0$ at the points (–1, –1) and (1, –3).

Therefore, the solutions of the quadratic equation $x^2 - x - 3 = 0$ and a line $x + y = -2$ are the points (−1, –1) and (1, –3).

Example 3

Find the equation of the given parabola:



Solution

Let the equation of parabola is

$y = ax^2 + bx + c$

Here, the points of parabola from the graph are A(–4, 0), B(2, 0), C(0, –8) and D(2, 0)

At point A(–4, 0), $0 = 16a - 4b + c$ ... (i)

At point B(2, 0), $0 = 4a + 2b + c$ ... (ii)

At point C(0, 8), $-8 = c$ ... (iii)

Now, from equation (i) and (iii)

$16a - 4b - 8 = 0$

or, $4(4a - b - 2) = 0$

or, $4a - b = 2$ ... (iv)

From equation (ii) and (iii)

$4a + 2b - 8 = 0$

or, $2(2a + b - 4) = 0$

or, $2a + b = 4$

or, $b = 4 - 2a$ ... (v)

Now, from equation (iv) and (v)

$4a - (4 - 2a) = 2$

or, $4a - 4 + 2a = 2$

or, $6a = 6$

or, $a = 1$

Now, put the value of a and c in equation (v)

$b = 4 - 2 \times 1 = 4 - 2 = 2$

Thus, $a = 1$, $b = 2$ and $c = -8$

Now, put the value of a, b and c in equation $y = ax^2 + bx + c$

$y = x^2 + 2x - 8$

Thus, required equation is $y = x^2 + 2x - 8$

Exercsie 4.2

1. Solve the given equations by graphical method.

a. $x^2 + 2x - 3 = 0$ b. $x^2 - 5x + 6 = 0$ c. $x^2 - 4x + 3 = 0$

d. $2x^2 - 7x + 3 = 0$ e. $x^2 - 3x - 10 = 0$

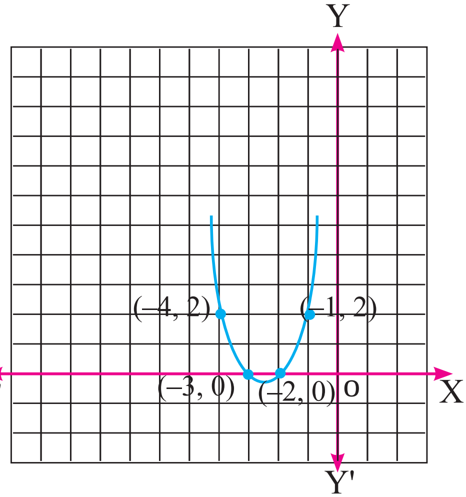
2. **Solve the given pair of equations by graphical method:**

a. $y = x^2$ and $y = x + 2$ b. $y = x^2 - x - 3$ and $x - y = 0$

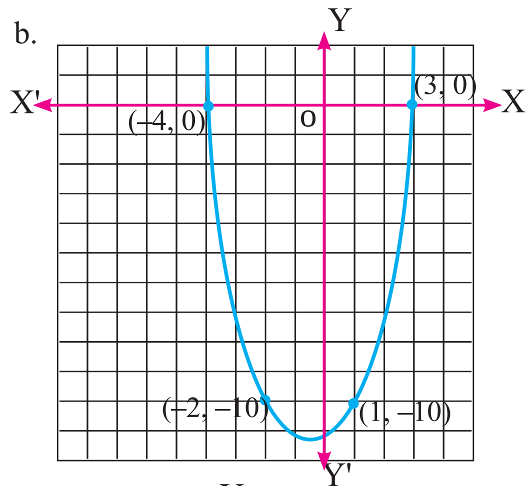
c. $y = 6x^2 - 2x - 15$ and $y = 4x - 3$ d. $y = x^2 - 2x$ and $y = x + 4$

3. **Find the equation of the given parabolas:**

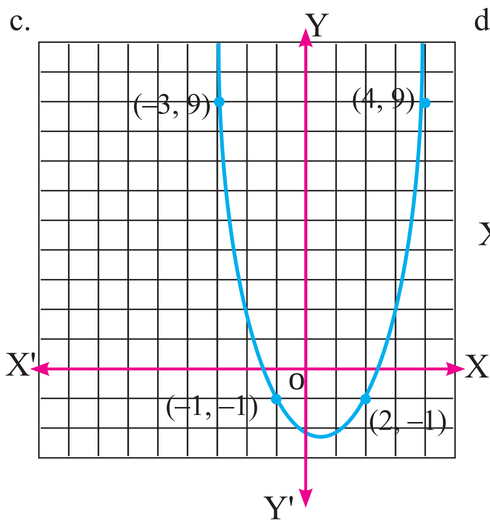
a.



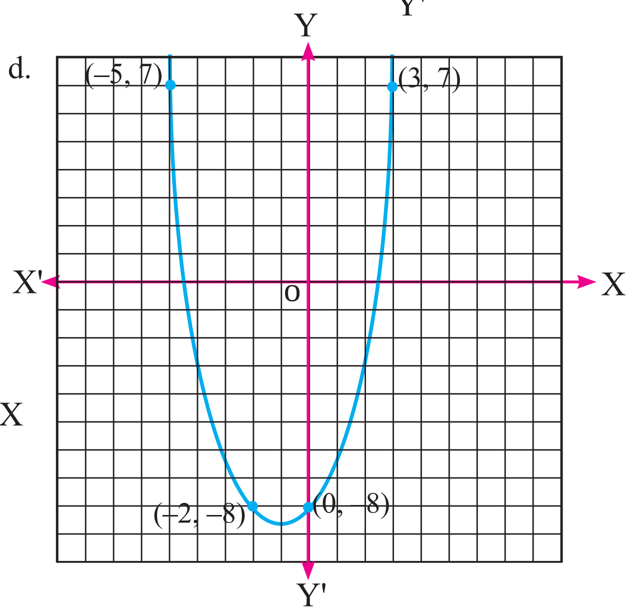
b.



c.



d.



Answer

1. a. (1, –3) b. (2, 3) c. (1, 3) d. $(\frac{1}{2}, 3)$ e. (–1, 5)

2. Show to the teacher.

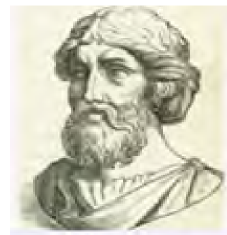
3. a. $y = x^2 + 5x + 6$ b. $y = x^2 + x - 12$ c. $y = x^2 - x - 3$ d. $y = x^2 + 2x - 8$

Lesson 5

Surd

5.1 Introduction

Numbers with a radical sign, whose value cannot be found in rational numbers, are called surds. Hippasus (500 BC) is considered the founder of the concept of surds, while the formal use of the term 'surd' was first done by Robert Recorde in 1551 AD. The values of numbers with radical signs: $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, $\sqrt{12}$, $\sqrt{45}$, $\sqrt[3]{2}$, $\sqrt[3]{5}$, etc., all yield rational numbers. Therefore, all these numbers are surds. Surds are of two types: pure surds and mixed surds. Pure surds are $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, $\sqrt[3]{2}$, $\sqrt[3]{5}$ while mixed surds are $\sqrt{12} = \sqrt{2 \times 2 \times 3} = 2\sqrt{3}$, $\sqrt{45} = \sqrt{3 \times 3 \times 5}$ $= 3\sqrt{5}$. Surds can be added, subtracted, multiplied and divided. Surds are used in engineering, physics, computer science, commerce, geometry, trigonometry, etc., to find values that cannot be represented in decimals and to reduce rounding errors.



Hippasus

5.2 Equations Involving Surds

Activity 1

Study the given questions, discuss and draw conclusions:

a. What could be the values of the x in the equations: $\sqrt{x} - 2 = 3$, $\sqrt{2x-1} = 3$ and $\sqrt{3x+10} - x = 0$?

b. How can equations involving surds be solved?

c. Is there any specific rule for solving equations?

Generally, to solve equations involving surds, the terms containing the radical sign and the terms not containing the radical sign must be separated. After that, square root or cube root in the equation can be squared or cubed to remove the radical sign and solve the equations involving surds.

Therefore, the following steps should be followed to solve equations involving surds.

Step 1: The terms containing the radical sign and the terms not containing the radical sign must be separated, for example: $\sqrt{x} - 2 = 3$ should be made into $\sqrt{x} = 5$.

Step 2: To remove the square root or cube root, both sides of the equation must be squared or cubed, for example: If $\sqrt{2x-1} = 3$ then, it should be made into $(\sqrt{2x-1})^2 = 3^2$.

Step 3: The obtained equation must be simplified and the new equation solved, for example: $2x - 1 = 9$ or, $2x = 9 + 1$ or, $2x = 10$ or, $x = 5$.

Step 4: When squaring or cubing equations involving surds, incorrect solutions may appear, so the obtained solutions must be checked by substituting them into the original equation. For example: When solving $\sqrt{3x + 10} - x = 0$ the values of x obtained are 5 and –2. Substituting $x = -2$ into the original equation makes the equation invalid, but substituting $x = 5$ makes the equation valid.

Step 5: If there are two terms with radical signs, the equation must be squared or cubed twice to eliminate the radical signs.

Example 1

Solve and verify:

a. $\sqrt{2x} = 6$ b. $\sqrt[3]{2x - 1} = 3$ c. $\sqrt{3x + 10} - x = 0$ d. $\sqrt{x + 1} - \sqrt{x} = 1$

Solution

a. Here, $\sqrt{2x} = 6$

Squaring on both sides

$(\sqrt{2x})^2 = 6^2$

or, $2x = 36$

or, $x = \frac{36}{2}$

$\therefore x = 18$

Verifying

Here, $\sqrt{2x} = 6$

Now, substituting $x = 18$

or, $\sqrt{2 \times 18} = 6$

or, $\sqrt{36} = 6$

or, $6 = 6$

Thus, $x = 18$

b. Here, $\sqrt[3]{2x - 1} = 3$

Cubing on both side

$(\sqrt[3]{2x - 1})^3 = 3^3$

or, $2x - 1 = 27$

or, $2x = 27 + 1$

or, $2x = 28$

or, $x = \frac{28}{2}$

$\therefore x = 14$

Verifying

Here, $\sqrt[3]{2x - 1} = 3$

Now, substituting $x = 14$

or, $\sqrt[3]{2 \times 14 - 1} = 3$

or, $\sqrt[3]{28 - 1} = 3$

or, $\sqrt[3]{27} = 3$

or, $3 = 3$

Thus, $x = 14$

c. Here, $\sqrt{3x+10} - x = 0$

or, $\sqrt{3x+10} = x$

Squaring on both sides

or, $(\sqrt{3x+10})^2 = (x)^2$

or, $3x + 10 = x^2$

or, $x^2 - 3x - 10 = 0$

or, $x^2 - x(5-2) - 10 = 0$

or, $x^2 - 5x + 2x - 10 = 0$

or, $x(x-5) + 2(x-5) = 0$

or, $(x+2)(x-5) = 0$

or, $x + 2 = 0$, Thus, $x = -2$

or, $x - 5 = 0$, Thus, $x = 5$

$\therefore$ $x = -2$ and 5

Verifying

Here, $\sqrt{3x+10} - x = 0$

Now, substituting $x = -2$

or, $\sqrt{3(-2)+10} - (-2) = 0$

or, $\sqrt{-6+10} + 2 = 0$

or, $\sqrt{4} + 2 = 0$

or, $2 + 2 = 0$

or, $4 \neq 0$, which is false. Thus, $x \neq -2$

Again, now, substituting $x = 5$

or, $\sqrt{3(5)+10} - (5) = 0$

or, $\sqrt{25} - 5 = 0$

or, $5 - 5 = 0$

or, $0 = 0$, which is true. Thus, $x = 5$

d. Here, $\sqrt{x+1} - \sqrt{x} = 1$

or, $\sqrt{x+1} = 1 + \sqrt{x}$

Squaring on both sides

or, $(\sqrt{x+1})^2 = (1 + \sqrt{x})^2$

or, $x + 1 = 1 + 2\sqrt{x} + x$

or, $0 = 2\sqrt{x}$

or, $\sqrt{x} = 0$

or, $x = 0$

$\therefore$ $x = 0$

Verifying

Here, $\sqrt{x+1} - \sqrt{x} = 1$

Now, substituting $x = 0$

$\sqrt{0+1} - \sqrt{0} = 1$

or, $\sqrt{1} = 1$

or, $1 = 1$, which is true. Thus, $x = 0$

Example 2

Solve and verify

$$\frac{x-1}{\sqrt{x}-1} = 3 + \frac{\sqrt{x}+1}{2}$$

Solution: Here,

$\frac{x-1}{\sqrt{x}-1} = 3 + \frac{\sqrt{x}+1}{2}$

or, $\frac{(\sqrt{x})^2-(1)^2}{\sqrt{x}-1} = \frac{6+\sqrt{x}+1}{2}$

or, $\frac{(\sqrt{x}+1)(\sqrt{x}-1)}{\sqrt{x}-1} = \frac{\sqrt{x}+7}{2}$

or, $\sqrt{x} + 1 = \frac{\sqrt{x}+7}{2}$

or, $2\sqrt{x} + 2 = \sqrt{x} + 7$

or, $2\sqrt{x} - \sqrt{x} = 7 - 2$

or, $\sqrt{x} = 5$

Squaring on both sides $(\sqrt{x})^2 = 5^2$

Thus, $x = 25$

Verifying

Here, $\frac{x-1}{\sqrt{x}-1} = 3 + \frac{\sqrt{x}+1}{2}$

Now, substituting $x = 25$

or, $\frac{25-1}{\sqrt{25}-1} = 3 + \frac{\sqrt{25}+1}{2}$

or, $\frac{24}{5-1} = 3 + \frac{5+1}{2}$

or, $\frac{24}{4} = 3 + 3$

or, $6 = 6$

Thus, $x = 25$

Example 3

Solve and verify: $\sqrt{x^2 - 3x + 5} - \sqrt{x^2 - 4x + 4} = 1$

Solution

Here, $\sqrt{x^2 - 3x + 5} - \sqrt{x^2 - 4x + 4} = 1$

or, $\sqrt{x^2 - 3x + 5} = 1 + \sqrt{x^2 - 4x + 4}$

or, $\sqrt{x^2 - 3x + 5} = 1 + \sqrt{(x-2)^2}$

or, $\sqrt{x^2 - 3x + 5} = 1 + x - 2$

or, $\sqrt{x^2 - 3x + 5} = x - 1$

Squaring on both sides

or, $(\sqrt{x^2 - 3x + 5})^2 = (x-1)^2$

or, $x^2 - 3x + 5 = x^2 - 2x + 1$

or, $x^2 - x^2 - 3x + 2x = 1 - 5$

or, $x^2 - x^2 - 3x + 2x = -4$

or, $-x = -4$

Thus $x = 4$

Verifying

Here, $\sqrt{x^2 - 3x + 5} - \sqrt{x^2 - 4x + 4} = 1$

Now, substituting $x = 4$

or, $\sqrt{4^2 - 3.4 + 5} - \sqrt{4^2 - 4.4 + 4} = 1$

or, $\sqrt{16 - 12 + 5} - \sqrt{16 - 16 + 4} = 1$

or, $\sqrt{9} - \sqrt{4} = 1$

or, $3 - 2 = 1$

or, $1 = 1$

Thus, $x = 4$

Exercie 5.1

1. Solve the given equations and verify.

a. $\sqrt{2x} = 6$ b. $\sqrt{3x - 5} = 1$ c. $6 - \sqrt{3x + 4} = 1$

d. $\sqrt[3]{5x - 3} = 3$ e. $\sqrt{3x + 6} - 3\sqrt{x - 4} = 0$

f. $\sqrt{11x^2 + 45} = 4x$

2. Solve the given equations and verify.

a. $\sqrt{x + 7} + \sqrt{x} = 7$ b. $\sqrt{x + 5} = 5 - \sqrt{x}$ c. $\sqrt{3x - 2} - \sqrt{x - 2} = 2$

d. $\sqrt{3x + 10} - x = 0$ e. $\sqrt{3x - 5} = x - 1$ f. $\sqrt{2x^2 - 7} = x + 3$

3. Solve the given equations and verify.

a. $\sqrt{x - 1} + \sqrt{x} = \frac{2}{\sqrt{x}}$ b. $\frac{x - 1}{\sqrt{x} + 1} = 3 + \frac{\sqrt{x} + 1}{2}$

c. $\sqrt{x^2 - 3x + 5} - \sqrt{x^2 - 4x + 4} = 1$ d. $\frac{\sqrt{x + 4} + \sqrt{x - 4}}{\sqrt{x + 4} - \sqrt{x - 4}} = 2$

e. $\frac{1}{\sqrt{x + 2}} - \frac{1}{\sqrt{4x + 8}} = 2$ f. $\sqrt{x^2 - 3x + 3} + \sqrt{x^2 - x + 1} = 2$

g. $\sqrt{x^2 - 2x - 4} - \sqrt{x^2 - 3x - 3} = 1$

4. Prove that the value of x is 81 while solving the equation involving surd

$\frac{x - 1}{\sqrt{x} + 1} = 4 + \frac{\sqrt{x} - 1}{2}$.

5. Construct any five question as given in the exercise and solve, then, present in classroom.

Answer

1. a. 18	b. 2	c. 7	d. 6	e. 7	f. 3, but not –3
2. a. 9	b. 4	c. 2, 6	d. 5, but not –2	e. 2, 3	f. –2, 8
3. a. $\frac{4}{3}$	b. 25	c. 4	d. 3	e. $\frac{-31}{16}$	f. 1
g. 4, but not $\frac{-4}{3}$	4 - 5. Show to the teacher.				

Lesson 6

Matrix and Determinant

6.1 Introduction

The rectangular arrangement of numbers in the form of rows and columns is called a matrix. The term matrix was introduced by James Joseph Sylvester, and the use of capital English letters such as A, B, … to denote matrices was developed by Arthur Cayley in 1858 AD. Similarly, during the development of determinants, the methods found in the unpublished works of Gottfried Wilhelm Leibniz were later developed into definite rules by Gabriel Cramer.

James Joseph Sylvester

Matrices are widely used in the fields of computer science, engineering, economics, statistics, etc. They make data processing and calculation faster and more efficient. Determinants are used in calculating area and volume.

6.2 Transpose of Matrix

Activity 1

Based on the matrix $A = \begin{bmatrix} 3 & -5 & 7 \\ 2 & 4 & -6 \end{bmatrix}$**, discuss and answer the following questions.**

a. How many rows are there in matrix A?

b. How many columns are there in matrix A?

c. What is the order of matrix A?

d. Write the matrix obtained when the rows of A are written as columns and the columns as rows.

e. Write the order of the new matrix obtained in (d).

Matrix A has two rows and three columns. Therefore, the order of matrix A is 2 × 3.

When the numbers in the rows of matrix A are written as columns and the numbers in the columns are written as rows, the new matrix becomes

$\begin{bmatrix} 3 & 2 \\ -5 & 4 \\ 7 & -6 \end{bmatrix}$ where the order is 3 × 2. The transpose of the given matrix A is denoted by A', A^t, or A^T. Therefore, $A' = \begin{bmatrix} 3 & 2 \\ -5 & 4 \\ 7 & -6 \end{bmatrix}$.

The new matrix is obtained by interchanging the rows into columns and columns into rows of a matrix is called the transpose of the matrix. If the given matrices are A, B, C, … then, their transpose matrices are denoted by A', B', C', …. respectively.

6.1.1 Properties of Transpose of Matrix

Let A and B be two matrices of the same order and let k be any real number. The main properties of the transpose of a matrix are as follows:

a. Transpose of the sum of two matrices

Example: $A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$ then,

$$(A + B) = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 2+1 & 0+2 \\ 1+0 & 3+4 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 7 \end{bmatrix}$$

$$(A + B)^T = \begin{pmatrix} 3 & 1 \\ 2 & 7 \end{pmatrix}$$

Similarly, $A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$ then, $A^T = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$

and $B = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$ then, $B^T = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}$

Now, $A^T + B^T = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2+1 & 1+0 \\ 0+2 & 3+4 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 7 \end{bmatrix}$

Thus, $(A + B)^T = A^T + B^T = \begin{bmatrix} 3 & 1 \\ 2 & 7 \end{bmatrix}$

b. Transpose of the difference of two matrices

Let, $A = \begin{bmatrix} 5 & 3 \\ 6 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -7 & 2 \\ 4 & 8 \end{bmatrix}$ then,

$$A - B = \begin{bmatrix} 5 & 3 \\ 6 & -2 \end{bmatrix} - \begin{bmatrix} -7 & 2 \\ 4 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 5+7 & 3-2 \\ 6-4 & -2-8 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 1 \\ 2 & -10 \end{bmatrix}$$

$$(A - B)^T = \begin{bmatrix} 12 & 2 \\ 1 & -10 \end{bmatrix}$$

Again, $A^T = \begin{bmatrix} 5 & 6 \\ 3 & -2 \end{bmatrix}$ and $B^T = \begin{bmatrix} -7 & 4 \\ 2 & 8 \end{bmatrix}$

$A^T - B^T = \begin{bmatrix} 5 & 6 \\ 3 & -2 \end{bmatrix} - \begin{bmatrix} -7 & 4 \\ 2 & 8 \end{bmatrix} = \begin{bmatrix} 5+7 & 6-4 \\ 3-2 & -2-8 \end{bmatrix} = \begin{bmatrix} 12 & 2 \\ 1 & -10 \end{bmatrix}$

Thus, $(A - B)^T = A^T - B^T = \begin{pmatrix} 12 & 2 \\ 1 & -10 \end{pmatrix}$

c. Transpose of the scaler multiplication

Let, matrix $A = \begin{bmatrix} 2 & 3 \\ 8 & -3 \\ 5 & 1 \end{bmatrix}$ and $k = 2$, a scaler number where,

$kA = 2\begin{bmatrix} 2 & 3 \\ 8 & -3 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 16 & -6 \\ 10 & 2 \end{bmatrix}$

Again, $(kA)^T = \begin{bmatrix} 4 & 16 & 10 \\ 6 & -6 & 2 \end{bmatrix}$

Again, $A^T = \begin{bmatrix} 2 & 8 & 5 \\ 3 & -3 & 1 \end{bmatrix}$ and $kA^T = 2\begin{bmatrix} 2 & 8 & 5 \\ 3 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 16 & 10 \\ 6 & -6 & 2 \end{bmatrix}$

Thus, $(kA)^T = kA^T$

d. Transpose of the transpose of a matrix

The transpose of matrix A is A^T. Again, taking the transpose of A^T, we obtain the original matrix A.

For example, let a 2 × 2 matrix be $A = \begin{bmatrix} -1 & 4 \\ 7 & -3 \end{bmatrix}$ then, $A^T = \begin{bmatrix} -1 & 7 \\ 4 & -3 \end{bmatrix}$

Again, $(A^T)^T = \begin{bmatrix} -1 & 7 \\ 4 & -3 \end{bmatrix}^T = \begin{bmatrix} -1 & 4 \\ 7 & -3 \end{bmatrix} = A$

Thus, $(A^T)^T = A$

e. Transpose of a zero matrix

If O is a zero matrix, for example a 2 × 2 zero matrix, $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Taking its transpose, $O^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ Thus, $O^T = O$. Similarly, discuss on zero matrices of different orders, $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ find O^T.

f. Transpose of the identity (unit) matrix

I is an identity (unit) matrix. For example, a 2 × 2 identity matrix is $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Taking its transpose, $I^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Therefore, $(I)^T = I$ Similarly, find I^T and discuss for a 3×3 identity matrix $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

g. Transpose of a symmetric matrix

Let A be a symmetric matrix. Taking its transpose gives matrix A^T. In this case, there is no change between matrix A and matrix A^T.
For example,

$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ then, $A^T = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

Similarly, $B = \begin{bmatrix} 5 & 2 & 3 \\ 2 & 4 & 8 \\ 3 & 8 & 1 \end{bmatrix}$ then, $B^T = \begin{bmatrix} 5 & 2 & 3 \\ 2 & 4 & 8 \\ 3 & 8 & 1 \end{bmatrix}$

Example 1

If $M = \begin{bmatrix} -2 & 8 \\ -3 & 5 \end{bmatrix}$ find M^T.

Solution

Here, the given matrix $M = \begin{bmatrix} -2 & 8 \\ -3 & 5 \end{bmatrix}$

Therefore, $M' = \begin{bmatrix} -2 & -3 \\ 8 & 5 \end{bmatrix}$

Example 2

If $B = \begin{bmatrix} 7 & 6 \\ 5 & 4 \end{bmatrix}$ Prove that: $(B^T)^T = B$

Solution

Here, the given matrix $B = \begin{bmatrix} 7 & 6 \\ 5 & 4 \end{bmatrix}$

$B^T = \begin{bmatrix} 7 & 5 \\ 6 & 4 \end{bmatrix}$

Again, $(B^T)^T = \begin{bmatrix} 7 & 5 \\ 6 & 4 \end{bmatrix}^T = \begin{bmatrix} 7 & 6 \\ 5 & 4 \end{bmatrix} = B$

Hence, $(B^T)^T = B$ proved.

Example 3

If $A = \begin{bmatrix} 4 & 6 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & -1 \\ 3 & 7 \end{bmatrix}$ then,

a. Proved that: $(A + B)^T = A^T + B^T$

b. Can we say (A + B) is symmetric matrix? Give reason.

Solution

Here, given matrices are, $A = \begin{bmatrix} 4 & 6 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & -1 \\ 3 & 7 \end{bmatrix}$

a. $A + B = \begin{bmatrix} 4 & 6 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} -2 & -1 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 4-2 & 6-1 \\ 2+3 & 1+7 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 5 & 8 \end{bmatrix}$

Again, $(A + B)^T = \begin{bmatrix} 2 & 5 \\ 5 & 8 \end{bmatrix}^T = \begin{bmatrix} 2 & 5 \\ 5 & 8 \end{bmatrix}$.........(i)

Similarly, $A^T = \begin{bmatrix} 4 & 6 \\ 2 & 1 \end{bmatrix}^T = \begin{bmatrix} 4 & 2 \\ 6 & 1 \end{bmatrix}$ and $B^T = \begin{bmatrix} -2 & -1 \\ 3 & 7 \end{bmatrix}^T = \begin{bmatrix} -2 & 3 \\ -1 & 7 \end{bmatrix}$

$A^T + B^T = \begin{bmatrix} 4 & 2 \\ 6 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 3 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 4-2 & 2+3 \\ 6-1 & 1+7 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 5 & 8 \end{bmatrix}$

$\therefore A^T + B^T = \begin{bmatrix} 2 & 5 \\ 5 & 8 \end{bmatrix}$ ....(ii)

From equation (i) and (ii), $(A + B)^T = A^T + B^T$ proved.

b. (A + B) is called symmetric matrix, because $(A + B) = (A + B)^T$

Example 4

If $A = \begin{bmatrix} 6 & 7 \\ 8 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 8 & -2 \\ 3 & 5 \end{bmatrix}$, $C = \begin{bmatrix} p & 11 \\ 5 & q \end{bmatrix}$ and $(A + B)^T = C$ then, find the values of *p* and *q*.

Solution

Here, $A = \begin{bmatrix} 6 & 7 \\ 8 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 8 & -2 \\ 3 & 5 \end{bmatrix}$, $C = \begin{bmatrix} p & 2 \\ 3 & q \end{bmatrix}$ and $(A + B)^T = C$

$(A + B) = \begin{bmatrix} 6 & 7 \\ 8 & 2 \end{bmatrix} + \begin{bmatrix} 8 & -2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 6+8 & 7-2 \\ 8+3 & 2+5 \end{bmatrix} = \begin{bmatrix} 14 & 5 \\ 11 & 7 \end{bmatrix}$

Again, $(A + B)^T = \begin{bmatrix} 14 & 5 \\ 11 & 7 \end{bmatrix}^T = \begin{bmatrix} 14 & 11 \\ 5 & 7 \end{bmatrix}$

According to the question, $(A + B)^T = C$

or, $\begin{bmatrix}14 & 11\\ 5 & 7\end{bmatrix} = \begin{bmatrix}p & 2\\ 3 & q\end{bmatrix}$

Equating the corresponding elements of equal matrices, $p = 14$ and $q = 7$

Thus, $p = 14$ and $q = 7$

Exercise 6.1

1. Tick (✓) the correct option for the given questions:

A. What is meant by the transpose of a matrix?

a. Only change of rows b. Only change of columns

c. Only change of signs d. Interchange of rows and columns

B. If $C = \begin{bmatrix}5 & -1 & -2\\ 4 & 6 & 8\end{bmatrix}$, then, which of the following is the matrix C^T?

a. $\begin{bmatrix}5 & -1 & -2\\ 4 & 6 & 8\end{bmatrix}$ b. $\begin{bmatrix}5 & 4\\ -1 & 6\\ -2 & 8\end{bmatrix}$ c. $\begin{bmatrix}4 & 5\\ 6 & -1\\ 8 & -2\end{bmatrix}$ d. $\begin{bmatrix}-5 & -4\\ 1 & -6\\ 2 & -8\end{bmatrix}$

C. If the symmetric matrix $A = \begin{bmatrix}5 & -3\\ -3 & 4\end{bmatrix}$, then, which of the following is A^T?

a. $\begin{bmatrix}5 & -3\\ -3 & 4\end{bmatrix}$ b. $\begin{bmatrix}-5 & -3\\ -3 & -4\end{bmatrix}$ c. $\begin{bmatrix}5 & 3\\ 3 & 4\end{bmatrix}$ d. $\begin{bmatrix}5 & -3\\ 4 & -3\end{bmatrix}$

D. If $P = \begin{bmatrix}0 & 3\\ 2 & 1\\ 4 & 7\end{bmatrix}$ and $Q = \begin{bmatrix}3 & -5\\ -6 & 1\\ 8 & 2\end{bmatrix}$, then, which of the following is $(P + Q)^T$?

a. $\begin{bmatrix}3 & -4 & 12\\ -2 & 2 & 9\end{bmatrix}$ b. $\begin{bmatrix}3 & -2\\ -4 & 2\\ 12 & 9\end{bmatrix}$ c. $\begin{bmatrix}-3 & 4 & -12\\ 2 & -2 & -9\end{bmatrix}$ d. $\begin{bmatrix}-3 & -4\\ 1 & -6\\ 2 & -8\end{bmatrix}$

2. Find the transpose matrices of the given matrices:

a. $R = \begin{bmatrix}3 & 2 & 1\end{bmatrix}$ b. $W = \begin{bmatrix}-3\\ 4\\ 5\end{bmatrix}$ c. $X = \begin{bmatrix}-1 & 5\\ 3 & 7\end{bmatrix}$

d. $Y = \begin{bmatrix}3 & 4 & 7\\ 2 & -1 & 10\end{bmatrix}$ e. $Z = \begin{bmatrix}0 & 1\\ 5 & -4\\ 4 & 5\end{bmatrix}$ f. $M = \begin{bmatrix}6 & 7 & 1\\ 0 & 5 & 2\\ -1 & 3 & 4\end{bmatrix}$

3. If $P = \begin{bmatrix}9 & 4\\ 8 & 5\end{bmatrix}$, prove that: $(P^T)^T = P$

4. If $M = \begin{bmatrix} p & q & r \\ a & b & c \\ x & y & z \end{bmatrix}$, then,

a. Find M and $(M^T)^T$ b. Write the relation of M and $(M^T)^T$

5. If $A = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -3 \\ 2 & 5 \end{bmatrix}$, then,

a. Find $(A^T + B^T)$ b. Find $(A + B)^T$

c. Can $(A + B)^T$ and $(A^T + B^T)$ be written equal? Give reason.

6. If $A = \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$, then,

a. Prove that $(A + B)^T = A^T + B^T$

b. Can (A + B) be called a symmetric matrix? Give reason.

7. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} 4 & -3 \\ -2 & -5 \end{bmatrix}$ and $A^T = B$, find the values of a, b, c and d.

Answer

1. A. d B. b C. a D. a

2. a. $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ b. $\begin{bmatrix} -3 & 4 & 5 \end{bmatrix}$ c. $\begin{bmatrix} -1 & 3 \\ 5 & 7 \end{bmatrix}$ d. $\begin{bmatrix} 3 & 2 \\ 4 & -1 \\ 7 & 10 \end{bmatrix}$ e. $\begin{bmatrix} 0 & 5 & 4 \\ 1 & -4 & 5 \end{bmatrix}$ f. $M = \begin{bmatrix} 6 & 0 & -1 \\ 7 & 5 & 3 \\ 1 & 2 & 4 \end{bmatrix}$

3. Show to the teacher. 4. a. $M' = \begin{bmatrix} p & a & x \\ q & b & y \\ r & c & z \end{bmatrix}$ and $(M')' = \begin{bmatrix} p & q & r \\ a & b & c \\ x & y & z \end{bmatrix}$ b. equal

5. a. $\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$ c. Yes

6. a. Show to the teacher b. No 7. $a = 4$, $c = -3$, $b = -2$, $d = -5$

6.2 Multiplication of Matrix

Activity 1

Answer the following questions by discussing on the basis of matrices $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -3 \\ -2 & -5 \end{bmatrix}$.

a. What are the orders of matrices A and B?

b. How can the matrix obtained by multiplying matrix A and matrix B be found?

c. Are AB and BA equal?

The order of the given matrix A is 2 × 2, and the order of matrix B is also 2 × 2.

Now, to find AB, multiply the rows of matrix A with the columns of matrix B as follows:

$$AB = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -2 & -5 \end{bmatrix} = \ldots\ldots?$$

1. **First step:** first row × first column

$$\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -2 & -5 \end{bmatrix} = \begin{bmatrix} 3 \times 1 + 4 \times (-2) & \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \end{bmatrix}$$

2. **Second step:** first row × second column

$$\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -2 & -5 \end{bmatrix} = \begin{bmatrix} 3 \times 1 + 4 \times (-2) & 3 \times (-3) + 4 \times (-5) \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \end{bmatrix}$$

3. **Third step:** second row × first column

$$\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -2 & -5 \end{bmatrix} = \begin{bmatrix} 3 \times 1 + 4 \times (-2) & 3 \times (-3) + 4 \times (-5) \\ 5 \times 1 + 6 \times (-2) & \ldots\ldots\ldots\ldots \end{bmatrix}$$

4. **Fourth step:** second row × second column

$$\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -2 & -5 \end{bmatrix} = \begin{bmatrix} 3 \times 1 + 4 \times (-2) & 3 \times (-3) + 4 \times (-5) \\ 5 \times 1 + 6 \times (-2) & 5 \times (-3) + 6 \times (-5) \end{bmatrix}$$

$$= \begin{bmatrix} 3 - 8 & -9 - 20 \\ 5 - 12 & -15 - 30 \end{bmatrix} = \begin{bmatrix} -5 & -29 \\ -7 & -45 \end{bmatrix}$$

Thus, from the above steps, $AB = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -2 & -5 \end{bmatrix} = \begin{bmatrix} -5 & -29 \\ -7 & -45 \end{bmatrix}_{2 \times 2}$

Similarly, find BA. Are AB and BA equal?

In two matrices A and B, if the number of columns of the first matrix A is equal to the number of rows of the second matrix B, then, matrix multiplication is possible. The product of matrices A and B is denoted by AB.

If the orders of matrices A and B are respectively m×n and n×p, then, the order of the product AB is m×p.

Symbolically,

$$A_{m \times n} \times B_{n \times p} = (AB)_{m \times p}$$

If the number of columns of the first matrix is equal to the number of rows of the second matrix, then, matrix multiplication is possible; otherwise multiplication is not possible.

Thought Provoking Question: The number of columns of matrix B is p and the number of rows of matrix A is m. Is it possible to find BA?

6.2.1 Properties Related to Matrix Multiplication

1. **Commutative property :** Generally, the commutative law is not true for matrix multiplication. If A and B are two matrices, then AB ≠ BA.

2. **Associative property:** If A, B, and C are three matrices, then whenever the multiplication is possible, (AB)C = A(BC)

If $A = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -1 \\ -2 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 3 \\ 6 & 0 \end{bmatrix}$

$$AB = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 2\times 4 + 0\times(-2) & 2\times(-1) + 0\times 5 \\ 3\times 4 + 1\times(-2) & 3\times(-1) + 1\times 5 \end{bmatrix}$$

$$= \begin{bmatrix} 8+0 & -2+0 \\ 12-2 & -3+5 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -2 \\ 10 & 2 \end{bmatrix}$$

$$BC = \begin{bmatrix} 4 & -1 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 6 & 0 \end{bmatrix} = \begin{bmatrix} 4\times 1 + (-1)\times 6 & 4\times 3 + (-1)\times 0 \\ (-2)\times 1 + 5\times 6 & (-2)\times 3 + 5\times 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4-6 & 12-0 \\ -2+30 & -6+0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 12 \\ 28 & -6 \end{bmatrix}$$

$$(AB)\,C = \begin{bmatrix} 8 & -2 \\ 10 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 6 & 0 \end{bmatrix} = \begin{bmatrix} 8\times 1 + (-2)\times 6 & 8\times 3 + (-2)\times 0 \\ 10\times 1 + 2\times 6 & 10\times 3 + 2\times 0 \end{bmatrix}$$

$$= \begin{bmatrix} 8-12 & 24+0 \\ 10+12 & 30+0 \end{bmatrix}$$

$$\therefore (AB)\,C = \begin{bmatrix} -4 & 24 \\ 22 & 30 \end{bmatrix} \text{..............................(i)}$$

Similarly,

$$A(BC) = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -2 & 12 \\ 28 & -6 \end{bmatrix} = \begin{bmatrix} 2\times(-2) + 0\times 28 & 2\times 12 + 0\times(-6) \\ 3\times(-2) + 1\times 28 & 3\times 12 + 1\times(-6) \end{bmatrix}$$

$$= \begin{bmatrix} -4+0 & 24+0 \\ -6+28 & 36-6 \end{bmatrix}$$

$$\therefore A(BC) = \begin{bmatrix} -4 & 24 \\ 22 & 30 \end{bmatrix} \text{..............................(i)}$$

Thus, from equations (i) and (ii), (AB) C = A(BC)

3. **Distributive property:** If A, B and C are three matrices, then whenever the multiplication of A(B + C) and (AB + AC) are possible, then,
A(B + C) = AB + AC.

For example: $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & -3 \\ 6 & 0 \end{bmatrix}$ then,

$(B + C) = \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} -1 & -3 \\ 6 & 0 \end{bmatrix} = \begin{bmatrix} 1-1 & 4-3 \\ 2+6 & 5+0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 8 & 5 \end{bmatrix}$

$A\,(B + C) = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 8 & 5 \end{bmatrix} = \begin{bmatrix} 1\times0+2\times8 & 1\times1+2\times5 \\ 0\times0+3\times8 & 0\times1+3\times5 \end{bmatrix}$

$= \begin{bmatrix} 0+16 & 1+10 \\ 0+24 & 0+15 \end{bmatrix}$

$\therefore A\,(B + C) = \begin{bmatrix} 16 & 11 \\ 24 & 15 \end{bmatrix}$..........................(i)

Again, $AB = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1\times1+2\times2 & 1\times4+2\times5 \\ 0\times1+3\times2 & 0\times4+3\times5 \end{bmatrix}$

$= \begin{bmatrix} 1+4 & 4+10 \\ 0+6 & 0+15 \end{bmatrix}$

$= \begin{bmatrix} 5 & 14 \\ 6 & 15 \end{bmatrix}$

Simalarly, $AC = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & -3 \\ 6 & 0 \end{bmatrix} = \begin{bmatrix} 1\times(-1)+2\times6 & 1\times(-3)+2\times0 \\ 0\times(-1)+3\times6 & 0\times(-3)+3\times0 \end{bmatrix}$

$= \begin{bmatrix} -1+12 & -3+0 \\ 0+18 & 0+0 \end{bmatrix}$

$= \begin{bmatrix} 11 & -3 \\ 18 & 0 \end{bmatrix}$

$AB + AC = \begin{bmatrix} 5 & 14 \\ 6 & 15 \end{bmatrix} + \begin{bmatrix} 11 & -3 \\ 18 & 0 \end{bmatrix} = \begin{bmatrix} 5+11 & 14-3 \\ 6+18 & 15+0 \end{bmatrix} = \begin{bmatrix} 16 & 11 \\ 24 & 15 \end{bmatrix}$

$\therefore AB + AC = \begin{bmatrix} 16 & 11 \\ 24 & 15 \end{bmatrix}$....................................(ii)

Thus, from equations (i) and (ii), A (B + C) = AB + AC

4. **Identity property of matrix multiplication**

If A be square matrix and I be an identity matrix of same order, then AI = IA = A.

Example:

$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then,

$$AI = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2\times 1 + 3\times 0 & 2\times 0 + 3\times 1 \\ 1\times 1 + 4\times 0 & 1\times 0 + 4\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2+0 & 0+3 \\ 1+0 & 0+4 \end{bmatrix}$$

$$\therefore AI = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} = A \text{..........................(i)}$$

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1\times 2 + 0\times 1 & 1\times 3 + 0\times 4 \\ 0\times 2 + 1\times 1 & 0\times 3 + 1\times 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2+0 & 3+0 \\ 0+1 & 0+4 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} = A$$

$$\therefore IA = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} = A \text{..........................(ii)}$$

Thus, from equations (i) and (ii), AI = IA = A

Example 1

If $A = \begin{bmatrix} 2 & 4 & 0 \\ 3 & 5 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 4 \\ 8 & 2 \\ 3 & 7 \end{bmatrix}$ then,

a. Is AB defined? Give reason. If yes, find AB.

b. Is BA defined? Give reason. If yes, find BA.

Solution: Here,

$A = \begin{bmatrix} 2 & 4 & 0 \\ 3 & 5 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 4 \\ 8 & 2 \\ 3 & 7 \end{bmatrix}$ Where, the order of A is 2×3 and order of B is 3×2.

a. **For AB:** The number of columns of the first matrix A is equal to the number of rows of the second matrix B. Therefore, AB is defined.

$$\text{Now, } AB = \begin{bmatrix} 2 & 4 & 0 \\ 3 & 5 & 1 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 8 & 2 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 2\times(-1) + 4\times 8 + 0\times 3 & 2\times 4 + 4\times 2 + 0\times 7 \\ 3\times(-1) + 5\times 8 + 1\times 3 & 3\times 4 + 5\times 2 + 1\times 7 \end{bmatrix}$$

$$= \begin{bmatrix} -2+32+0 & 8+8+0 \\ -3+40+3 & 12+10+7 \end{bmatrix}$$

$$= \begin{bmatrix} 30 & 16 \\ 40 & 29 \end{bmatrix}$$

$$\text{Thus, } AB = \begin{bmatrix} 30 & 16 \\ 40 & 29 \end{bmatrix}$$

b. **For BA:** The number of columns of the first matrix B is equal to the number of rows of the second matrix A. Therefore, BA is defined.

Now, BA= $\begin{bmatrix} -1 & 4 \\ 8 & 2 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} 2 & 4 & 0 \\ 3 & 5 & 1 \end{bmatrix} = \begin{bmatrix} (-1)\times 2 + 4\times 3 & (-1)\times 4 + 4\times 5 & (-1)\times 0 + 4\times 1 \\ 8\times 2 + 2\times 3 & 8\times 4 + 2\times 5 & 8\times 0 + 2\times 1 \\ 3\times 2 + 7\times 3 & 3\times 4 + 7\times 5 & 3\times 0 + 7\times 1 \end{bmatrix}$

$$= \begin{bmatrix} -2+12 & -4+20 & 0+4 \\ 16+6 & 32+10 & 0+2 \\ 6+21 & 12+35 & 0+7 \end{bmatrix} = \begin{bmatrix} 10 & 16 & 4 \\ 22 & 42 & 2 \\ 27 & 47 & 7 \end{bmatrix}$$

Thus, BA = $\begin{bmatrix} 10 & 16 & 4 \\ 22 & 42 & 2 \\ 27 & 47 & 7 \end{bmatrix}$

Example 2

If A = $\begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$, then, prove that: $A^2 - 6A + 9I = O$ where, I and O are identity and zero matrix of order 2 × 2.

Solution: Here,

A = $\begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$, 2 × 2 identity matrix (I) = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and zero matrix (O) = $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Now, $A^2 = A \times A = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} \times \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 4\times 4 + 1\times(-1) & 4\times 1 + 1\times 2 \\ (-1)\times 4 + 2\times(-1) & (-1)\times 1 + 2\times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 16-1 & 4+2 \\ -4-2 & -1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 6 \\ -6 & 3 \end{bmatrix}$$

> Note: A^2 is not the square of all members, matrix A is multiplied by matrix A.

$$6A = 6\begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 24 & 6 \\ -6 & 12 \end{bmatrix}$$

$$9I = 9\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$$

L.H.S. = $A^2 - 6A + 9I = \begin{bmatrix} 15 & 6 \\ -6 & 3 \end{bmatrix} - \begin{bmatrix} 24 & 6 \\ -6 & 12 \end{bmatrix} + \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$

$$= \begin{bmatrix} 15-24+9 & 6-6+0 \\ -6+6+0 & 3-12+9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

= O = R.H.S.

Thus, $A^2 - 6A + 9I = O$ proved.

Example 3

If $A = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}$, $B = \begin{bmatrix} p & q \\ 0 & r \end{bmatrix}$ and AB = A + B, find the values of p, q and r

Solution: Here,

$A = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}$, $B = \begin{bmatrix} p & q \\ 0 & r \end{bmatrix}$ and AB = A + B

$$AB = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} p & q \\ 0 & r \end{bmatrix} = \begin{bmatrix} 4 \times p + 0 \times 0 & 4 \times q + 0 \times r \\ 0 \times p + 5 \times 0 & 0 \times q + 5 \times r \end{bmatrix}$$

$$= \begin{bmatrix} 4p + 0 & 4q + 0 \\ 0 + 0 & 0 + 5r \end{bmatrix} = \begin{bmatrix} 4p & 4q \\ 0 & 5r \end{bmatrix}$$

$$A + B = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix} + \begin{bmatrix} p & q \\ 0 & r \end{bmatrix} = \begin{bmatrix} 4 + p & 0 + q \\ 0 + 0 & 5 + r \end{bmatrix} = \begin{bmatrix} 4 + p & q \\ 0 & 5 + r \end{bmatrix}$$

According to questions, AB = A + B

or, $\begin{bmatrix} 4p & 4q \\ 0 & 5r \end{bmatrix} = \begin{bmatrix} 4 + p & q \\ 0 & 5 + r \end{bmatrix}$

Equating the corresponding values of equal matrices,

or, $4p = 4 + p$
or, $4p - p = 4$
or, $3p = 4$
or, $p = \frac{4}{3}$

or, $4q = q$
or, $4q - q = 0$
or, $3q = 0$
or, $q = 0$

or, $5r = 5 + r$
or, $5r - r = 5$
or, $4r = 5$
or, $r = \frac{5}{4}$

Thus, $p = \frac{4}{3}$, $q = 0$ and $r = \frac{5}{4}$

Example 4

If $A = \begin{bmatrix} 1 & 2 \\ 0 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$, prove that: $(AB)^T = B^T . A^T$

Solution: Here,

$A = \begin{bmatrix} 1 & 2 \\ 0 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 9 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \times 3 + 2 \times 2 & 1 \times 4 + 2 \times 1 \\ 0 \times 3 + 9 \times 2 & 0 \times 4 + 9 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 + 4 & 4 + 2 \\ 0 + 18 & 0 + 9 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 18 & 9 \end{bmatrix}$$

$(AB)^T = \begin{bmatrix} 7 & 6 \\ 18 & 9 \end{bmatrix}^T$

$\therefore (AB)^T = \begin{bmatrix} 7 & 18 \\ 6 & 9 \end{bmatrix}$ ....(i)

Similarly, $A^T = \begin{bmatrix} 1 & 2 \\ 0 & 9 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 2 & 9 \end{bmatrix}$ and $B^T = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}^T = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$

$B^T.A^T = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 9 \end{bmatrix} = \begin{bmatrix} 3 \times 1 + 2 \times 2 & 3 \times 0 + 2 \times 9 \\ 4 \times 1 + 1 \times 2 & 4 \times 0 + 1 \times 9 \end{bmatrix}$

$= \begin{bmatrix} 3 + 4 & 0 + 18 \\ 4 + 2 & 0 + 9 \end{bmatrix}$

$\therefore B^T.A^T = \begin{bmatrix} 7 & 18 \\ 6 & 9 \end{bmatrix}$ ....(ii)

Thus, from equations (i) and (ii), $(AB)^T = B^T.A^T$ proved.

Exercise 6.2

1. Tick mark (✓) the correct option for the given equestions:

A. Under which condition is the multiplication of two matrices possible?
   a. When the number of columns of the first matrix is equal to the number of rows of the second matrix.
   b. When the number of rows of the first matrix is equal to the number of columns of the second matrix.
   c. When the number of rows of the first matrix is equal to the number of rows of the second matrix.
   d. When the number of columns of the first matrix is equal to the number of columns of the second matrix.

B. If A, B, and C are three matrices and (AB)C = A(BC), what is the name of this property?
   a. Commutative property b. Associative property
   c. Distributive property d. Identity property

C. If A, B, and C are three matrices and A(B + C) = AB + AC, what is the name of this property?
   a. Commutative property b. Associative property
   c. Distributive property d. Identity property

D. If A and I are respectively a square matrix and an identity matrix of the same order and AI = IA = A, what is the name of this property?
   a. Commutative property b. Associative property
   c. Distributive property d. Identity property

E. If P = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and I = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then, which of the following is the product PI?

a. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ c. $\begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix}$ d. $\begin{bmatrix} 2 & 3 \\ 2 & 5 \end{bmatrix}$

F. If W = $\begin{bmatrix} -3 \\ 4 \\ 5 \end{bmatrix}$ and R = $\begin{bmatrix} 3 & 2 & 1 \end{bmatrix}$, then, what is the order of matrix WR?

a. 3×1 b. 1×3 c. 2×2 d. 3×3

2. **In the following cases, find whether the multiplication of matrices is possible and determine the product if possible:**

a. $\begin{bmatrix} 5 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ b. $\begin{bmatrix} 1 \\ 3 \end{bmatrix}\begin{bmatrix} 5 & 4 \end{bmatrix}$ c. $\begin{bmatrix} 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ 4 \\ 5 \end{bmatrix}$

d. $\begin{bmatrix} 0 & 1 \\ 5 & -4 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 3 & 4 & 7 \\ 2 & -1 & 10 \end{bmatrix}$ e. $\begin{bmatrix} -1 & 5 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} 6 & 1 \\ 8 & 2 \end{bmatrix}$ f. $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -3 & 4 \\ 0 & 6 \\ 1 & 2 \end{bmatrix}$

3. If Q = $\begin{bmatrix} 3 & 6 \\ -2 & -4 \end{bmatrix}$, find Q^2

4. a. If M = $\begin{bmatrix} -2 & 3 \\ -1 & 2 \end{bmatrix}$ and N = $\begin{bmatrix} 4 & -3 \\ 5 & 0 \end{bmatrix}$, find MN and NM.

b. If C = $\begin{bmatrix} 3 & 5 \\ 0 & 1 \end{bmatrix}$, D = $\begin{bmatrix} -2 & 4 \\ -1 & 3 \end{bmatrix}$ and E = $\begin{bmatrix} 2 & -3 \\ 6 & 4 \end{bmatrix}$, find C(D + E)

5. a. If A = $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, prove that: $A^2 = 2A$

b. If X = $\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}$ and Y = $\begin{bmatrix} 2 & -5 \\ 2 & -5 \end{bmatrix}$, prove that XY is a zero matrix.

c. If R = $\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$, prove that: $R^2 = I$, where, I is a 2×2 identity matrix.

6. If A = $\begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$, prove that: $A^2 - A + 6I = O$ where I and O are respectively the identity matrix and zero matrix of order 2×2.

7. If U = $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$, V = $\begin{bmatrix} p & q \\ 0 & r \end{bmatrix}$, are two matrices and UV = U + V, find the values of p, q, and r.

8. If X = $\begin{bmatrix} 6 & 3 \\ -2 & 0 \end{bmatrix}$ and Y = $\begin{bmatrix} 2 & 5 \\ 1 & -4 \end{bmatrix}$, prove that $(XY)^T = Y^T.X^T$

9. If $A = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$, $C = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then,

a. Prove that $AB \neq BA$. (b) Prove that $A(BC) = (AB)C$.

c. Prove that $A(B + C) = AB + AC$.

d. Prove that $IA = AI = A$

e. Prove that $(AB)^T = B^T A^T$.

Answer

1. A. a B. b C. c D. d E. a F. d

2. a. [17] b. $\begin{bmatrix} 5 & 4 \\ 15 & 12 \end{bmatrix}$ c. [4] d. $\begin{bmatrix} 2 & -1 & 10 \\ 7 & 24 & -5 \\ 22 & 11 & 78 \end{bmatrix}$ e. $\begin{bmatrix} 34 & 9 \\ 74 & 17 \end{bmatrix}$

f. Show to the teacher. 3. $\begin{bmatrix} -3 & -6 \\ 2 & 4 \end{bmatrix}$ 4. a. $\begin{bmatrix} 7 & 6 \\ 6 & 3 \end{bmatrix}$ b. $\begin{bmatrix} -5 & 6 \\ -10 & 15 \end{bmatrix}$

5. Show to the teacher. 6. Show to the teacher.

7. a. $p = \frac{3}{2}$, $q = 0$, $r = \frac{4}{3}$ b. $a = 1$, $b = 2$ c. $b = 1$

8. a. $[-3 \quad 3]$ (b) $\begin{bmatrix} 2 & -2 \\ -2 & -2 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ 9. Show to the teacher.

6.3 Determinant of Matrix

Activity 1

Study the patterns of the given numbers and discuss the following questions:

a. $|6|$ b. $\begin{vmatrix} 2 & 4 \\ 5 & 9 \end{vmatrix}$

i. What is the form of the given numbers?

ii. What are the vertical lines used to enclose the number patterns called?

iii. What are the common features found in both number patterns?

Among the above number patterns, in (a) there is only one number. It has one row and one column. This is the number pattern of a square matrix of order 1×1.

In the second number pattern (b), there are two rows and two columns. It is the number form of a square matrix of order 2 × 2.

The vertical lines used to enclose each number pattern are the symbols of determinants.

The presentation of numbers arranged in rows and columns inside two vertical lines is called a determinant. Determinants can be obtained only from square matrices. A determinant is a scalar quantity. If A is a square matrix, the determinant of A is denoted by $|A|$.

6.3.1 Determinant of Order One Matrix

Let, A = [8] be a square matrix of order 1 × 1. Therefore, |A| = |8| = 8. Similarly, B = [−7] is also a square matrix of order 1 × 1. Hence, |B| = |−7| = −7.

If a square matrix has one row and one column, the determinant of such a matrix is called the determinant of a first-order matrix. For a square matrix of order 1 × 1, the determinant value is equal to the element itself.

Thought Provoking Questions : What is the absolute value of −7, i.e., |−7| = ? Are determinant and absolute value the same?

6.3.2 Determinant of Order Two Matrix

Let, A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a square matrix of order 2 × 2.

Discuss how its determinant is calculated and what its value is. The determinant of matrix A is denoted by D or |A|.

Matrix A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and the determinant of A is written as $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. where, *a*, *b*, *c*, and d are the elements of determinant A and ad−bc is called the expansion of |A|.

Similarly, if B = $\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$ then the determinant of matrix B is $|B| = \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} = 2 \times 1 - 1 \times 3 = 2 - 3 = -1$

If a square matrix has two rows and two columns, the determinant of such a matrix is called the determinant of order two. In a determinant, the elements of the main diagonal and the secondary diagonal are multiplied respectively, and the difference of these products gives the value of the determinant.

6.3.3 Singular Matrix

Let, A = $\begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$ be a square matrix of order 2 × 2. The determinant of matrix A is

$|A| = \begin{vmatrix} 2 & 6 \\ 1 & 3 \end{vmatrix} = 2 \times 3 - 6 \times 1 = 6 - 6$

$\therefore |A| = 0$

If the determinant of a square matrix is zero, such a matrix is called a singular matrix.

If A is a square matrix and |A| = 0, then A is a singular matrix.

6.3.4 Non-singular Matrix

Let, M = $\begin{bmatrix} 4 & 3 \\ 5 & 6 \end{bmatrix}$ be a square matrix of order 2 × 2. The determinant of matrix M is

$|M| = \begin{vmatrix} 4 & 3 \\ 5 & 6 \end{vmatrix} = 4 \times 6 - 3 \times 5 = 24 - 15$

Thus, $|M| = 9 \neq 0$

A square matrix whose determinant is not zero is called a non-singular matrix. It is also called a regular matrix. If A is a square matrix and $\|A\| \neq 0$, then, A is a non-singular matrix.

Example 1

If C = $|-12|$, find $|C|$.

Solution: Here,

Matrix C = $|-12|$

Now, by using formula, $|C| = |-12| = -12$

Thus, $|C| = -12$.

Example 2

If A = $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ find $|A|$

Solution: Here,

Metrix A = $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Now, by using formula, $|A| = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$

$= \cos\theta \times \cos\theta - (-\sin\theta) \times \sin\theta$

$= \cos^2\theta + \sin^2\theta = 1$

Thus, $|A| = 1$

Example 3

If A = $\begin{bmatrix} -1 & x \\ 3 & 4 \end{bmatrix}$ is a singular matrix, find the value of x.

Solution: Here,

Matrix A = $\begin{bmatrix} -1 & x \\ -3 & 6 \end{bmatrix}$ is a singular matrix.

Thus, |AB| = 0

or, $\begin{vmatrix} -1 & x \\ -3 & 6 \end{vmatrix} = 0$

or, $(-1) \times 6 - (-3) \times x = 0$

or, $-6 + 3x = 0$

or, $3x = 6$

or, $x = 2$

Thus, $x = 2$

Example 4

If $P = \begin{bmatrix} 5 & -3 \\ 2 & -4 \end{bmatrix}$ and $Q = \begin{bmatrix} 4 & 2 \\ -3 & 5 \end{bmatrix}$, verify $|PQ| = |P||Q|$.

Solution: Here,

$P = \begin{bmatrix} 5 & -3 \\ 2 & -4 \end{bmatrix}$ र $Q = \begin{bmatrix} 4 & 2 \\ -3 & 5 \end{bmatrix}$

$|P| = \begin{vmatrix} 5 & -3 \\ 2 & -4 \end{vmatrix} = 5 \times (-4) - (-3) \times 2 = -20 + 6 = -14$

$|Q| = \begin{vmatrix} 4 & 2 \\ -3 & 5 \end{vmatrix} = 4 \times 5 - 2 \times (-3) = 20 + 6 = 26$

$|P|\,|Q| = -14 \times 26$

$\therefore |P|\,|Q| = -364$ ....(i)

$$PQ = \begin{bmatrix} 5 & -3 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} 5 \times 4 + (-3) \times (-3) & 5 \times 2 + (-3) \times 5 \\ 2 \times 4 + (-4) \times (-3) & 2 \times 2 + (-4) \times 5 \end{bmatrix}$$

$$= \begin{bmatrix} 20 + 9 & 10 - 15 \\ 8 + 12 & 4 - 20 \end{bmatrix} = \begin{bmatrix} 29 & -5 \\ 20 & -16 \end{bmatrix}$$

$|PQ| = \begin{vmatrix} 29 & -5 \\ 20 & -16 \end{vmatrix} = 29 \times (-16) - 20 \times (-5) = -464 + 100$

$\therefore |PQ| = -364$..................................(ii)

From equation (i) and (ii), $|PQ| = |P|.|Q| = -364$

Exercise 6.3

1. **Tick (✓) the correct option for the given questions:**

A. If the matrix A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then, what is the value of |A| among the following?

a. $bc - ad$ b. $ad - bc$ c. $ac - bd$ d. $bd - ac$

B. If the matrix A = [−5], then, what is the value of |A| among the following?

a. 5 b. $\frac{1}{5}$ c. –5 d. 5^{-1}

C. If the matrix A = $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, then, what is the value of |A| among the following?

a. −12 b. −5 c. 5 d. 10

D. If $\begin{vmatrix} 2 & x \\ 1 & x \end{vmatrix}$ = 0, then, what is the value of x?

a. 2 b. 1 c. −2 d. 0

2. **Find the value of the following determinants:**

a. |2| b. |–7| c. $\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix}$

d. $\begin{vmatrix} sinA & -cosA \\ cosA & sinA \end{vmatrix}$ e. $\begin{vmatrix} x+y & x-y \\ x-y & x+y \end{vmatrix}$ f. $\begin{vmatrix} x+y & x-y \\ x+y & x-y \end{vmatrix}$

g. $\begin{vmatrix} 3 & -4 \\ -3 & 5 \end{vmatrix}$ h. $\begin{vmatrix} x+y & x-y \\ x^2+xy+y^2 & x^2-xy+y^2 \end{vmatrix}$

3. **Find the value of the determinants of the following matrices:**

a. [10] b. [–8] c. $\begin{bmatrix} x & y \\ y & x \end{bmatrix}$ d. $\begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix}$

e. $\begin{bmatrix} 10 & 5 \\ 4 & 2 \end{bmatrix}$ f. $\begin{bmatrix} -7 & 11 \\ 2 & -4 \end{bmatrix}$

4. **Find the value of *x* in the following cases:**

a. $\begin{vmatrix} 3 & x \\ 2 & 8 \end{vmatrix}$ = 20 b. $\begin{vmatrix} 5 & -2 \\ 2x & -3 \end{vmatrix}$ = 1 c. $\begin{vmatrix} 3x & 2 \\ -4 & 3 \end{vmatrix}$ = 8

d. $\begin{vmatrix} x & -2x \\ 3x & 4x \end{vmatrix}$ = 100

5. **If A = $\begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & -5 \\ 6 & 2 \end{bmatrix}$, find the determinants of the following matrices:**

a. A –3B b. 2A + B c. 3A –2B d. A + 2B

6. Prove that:

a. $\begin{vmatrix} x^2 & y^2 \\ y & x \end{vmatrix} = (x - y)(x^2 + xy + y^2)$ b. $\begin{vmatrix} x + y & y + z \\ z - y & x - y \end{vmatrix} = x^2 - z^2$

c. $\begin{vmatrix} a - 1 & a^2 - a + 1 \\ a + 1 & a^2 + a + 1 \end{vmatrix} = -2$ d. $\begin{vmatrix} 1 + x & 1 \\ 1 & 1 + y \end{vmatrix} = xy\,(1 + \frac{1}{x} + \frac{1}{y})$

7. If $M = \begin{bmatrix} 5 & 2 \\ 4 & 3 \end{bmatrix}$ and $N = \begin{bmatrix} 1 & -3 \\ 6 & 2 \end{bmatrix}$, then, verify: $|MN| = |N|\,|M|$

Answer

1. A. b	B. c	C. c	D. d	2. a. 2	b. 7	c.5	
d. 1	e. $4xy$	f. 0		g. 3	h. $2y^2$	3. a. 10	b. –8
c. $x^2 - y^2$	d. 10	e. 0	f. 6				
4. a. 2	b. 4	c. 0	d. $\pm\sqrt{10}$				
5. a. 289	b. –6		c. 74	d. 114			

6 – 7. Show to the teacher.

6.4 Inverse Matrix

Activity 1

Take two non–zero square matrices $A = \begin{bmatrix} 7 & 5 \\ 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -5 \\ -4 & 7 \end{bmatrix}$ and discuss the following questions and answer them:

a. Multiply A by B and find AB.

b. Multiply B by A and find BA.

c. If AB and BA are equal to the identity matrix $I_{2 \times 2}$, what kind of matrices are A and B?

Now, AB $= \begin{bmatrix} 7 & 5 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ -4 & 7 \end{bmatrix} = \begin{bmatrix} 7 \times 3 + 5 \times (-4) & 7 \times (-5) + 5 \times 7 \\ 4 \times 3 + 3 \times (-4) & 4 \times (-5) + 3 \times 7 \end{bmatrix}$

$= \begin{bmatrix} 21 - 20 & -35 + 35 \\ 12 - 12 & -20 + 21 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Again, BA $= \begin{bmatrix} 3 & -5 \\ -4 & 7 \end{bmatrix} \begin{bmatrix} 7 & 5 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 3 \times 7 + (-5) \times 4 & 3 \times 5 + (-5) \times 3 \\ (-4) \times 7 + 7 \times 4 & (-4) \times 5 + 7 \times 3 \end{bmatrix}$

$= \begin{bmatrix} 21 - 20 & 15 - 35 \\ -28 + 28 & -20 + 21 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

From the above calculations,

AB = BA = I. In this situation, matrices A and B are inverse matrices of each other.

Where, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a 2 × 2 identity matrix.

If for a square matrix A there exists another matrix B such that: AB = BA = I. (where I is an identity matrix), then, A and B are called inverse matrices of each other.

Here, $B = A^{-1}$ and $A = B^{-1}$. A matrix must be non-singular to have an inverse matrix.

6.4.1 Process of Finding the Inverse of Given 2 × 2 Matrix

Let, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then, $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$

Assume, $A^{-1} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$

According to the definition of inverse matrix, $A A^{-1} = I$

or, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

or, $\begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Thought Provoking Questions: Why must (ad − bc) not be 0? What will happen, if ad − bc = 0?

According to the rule of equal matrices,

$ap + br = 1$..............(i) $aq + bs = 0$..............(ii)

$cp + dr = 0$..............(iii) $cq + ds = 1$..............(iv)

Solving equations (i) and (iii), $p = \frac{d}{ad-bc}$ and $r = \frac{-c}{ad-bc}$

Solving equations (ii) and (iv), $q = \frac{-b}{ad-bc}$ and $s = \frac{a}{ad-bc}$

$$\therefore A^{-1} = \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \left[\because \text{Taking common } \frac{1}{ad-bc}\right]$$

$$\therefore A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

While finding the inverse matrix of a given matrix, interchange the diagonal elements and change the signs of the other elements, then, divide by the determinant.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then, } A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \text{ where, } |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$$

6.4.2 Properties of Inverse Matrix

Activity 1

If $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$, verify the following properties of inverse matrix:

a. $A.A^{-1} = A^{-1}.A = I$

b. $(AB)^{-1} = B^{-1}. A^{-1}$

c. $(A^{-1})^{-1} = A$

d. $(A^{T})^{-1} = (A^{-1})^{T}$

e. $(I)^{-1} = I$

f. $|A^{-1}| = \frac{1}{|A|}$

Example 1

If $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, then,

a. Find $|A|$.

b. Find A^{-1}.

Solution: Here,

$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, then,

a. According to the formula, $|A| = \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = 4 \times 2 - 3 \times 1 = 8 - 3 = 5$

b. Again, according to the formula, $A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & \frac{-3}{5} \\ \frac{-1}{5} & \frac{2}{5} \end{bmatrix}$

Thus, the inverse matrix of $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ is $\begin{bmatrix} \frac{4}{5} & \frac{-3}{5} \\ \frac{-1}{5} & \frac{2}{5} \end{bmatrix}$

Example 2

If the inverse matrix of $\begin{bmatrix} 2m & 7 \\ 5 & 9 \end{bmatrix}$ is given $\begin{bmatrix} 9 & n \\ -5 & 4 \end{bmatrix}$, find the values of m and n.

Solution

Let, $A = \begin{bmatrix} 2m & 7 \\ 5 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 9 & n \\ -5 & 4 \end{bmatrix}$

According to the question, since B is the inverse matrix of A, AB = I. Thus,

or, $\begin{bmatrix} 2m & 7 \\ 5 & 9 \end{bmatrix} \begin{bmatrix} 9 & n \\ -5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

or, $\begin{bmatrix} 2m \times 9 + 7 \times (-5) & 2m \times n + 7 \times 4 \\ 5 \times 9 + 9 \times (-5) & 5 \times n + 9 \times 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

or, $\begin{bmatrix} 18m - 35 & 2mn + 28 \\ 0 & 5n + 36 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

According to rule of equal matrices,

or, $18m - 35 = 1$
or, $18m = 1 + 35$
or, $m = \frac{36}{18} = 2$

or, $5n + 36 = 1$
or, $5n = 1 - 36$
or, $n = \frac{-35}{5} = -7$

Hence, $m = 2$ and $n = -7$

Example 3

If the inverse matrix of A is $A^{-1} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, find the matrix A.

Solution: Here,

Inverse matrix of A, $A^{-1} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

Let, $|A^{-1}| = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = 4 \times 1 - 3 \times 2 = 4 - 6 = -2$

Inverse matrix, $(A^{-1})^{-1} = \frac{1}{|A^{-1}|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$= \frac{1}{-2} \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}$$

$$\therefore (A^{-1})^{-1} = \begin{bmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}$$

We know that, $(A^{-1})^{-1} = A$

Thus, matrix $A = \begin{bmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}$

Example 4

If $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 7 \\ -3 & 9 \end{bmatrix}$, prove that: $(AB)^{-1} = B^{-1}. A^{-1}$

Solution: Here,

$A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 7 \\ -3 & 9 \end{bmatrix}$

According to the question,

$$AB = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} -2 & 7 \\ -3 & 9 \end{bmatrix} = \begin{bmatrix} 3\times(-2)+2\times(-3) & 3\times7+2\times9 \\ 7\times(-2)+5\times(-3) & 7\times7+5\times9 \end{bmatrix}$$

$$= \begin{bmatrix} -6-6 & 21+18 \\ -14-15 & 49+45 \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 39 \\ -29 & 94 \end{bmatrix}$$

$$|AB| = \begin{vmatrix} -12 & 39 \\ -29 & 94 \end{vmatrix} = -12\times94 - 39\times(-29) = -1128 + 1131 = 3$$

Again, by using the formula, $(AB)^{-1} = \dfrac{1}{|AB|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\therefore (AB)^{-1} = \dfrac{1}{3}\begin{bmatrix} 94 & -39 \\ 29 & -12 \end{bmatrix}$..........................(i)

$|A| = \begin{vmatrix} 3 & 2 \\ 7 & 5 \end{vmatrix} = 15 - 14 = 1$ and $|B| = \begin{vmatrix} -2 & 7 \\ -3 & 9 \end{vmatrix} = -18 + 21 = 3$

According to the formula, $A^{-1} = \dfrac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \dfrac{1}{1}\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$ and

$$B^{-1} = \frac{1}{|B|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 9 & -7 \\ 3 & -2 \end{bmatrix}$$

Now, $B^{-1}A^{-1} = \dfrac{1}{3}\begin{bmatrix} 9 & -7 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

$$= \frac{1}{3}\begin{bmatrix} 9\times5+(-7)(-7) & 9\times(-2)+(-7)\times3 \\ 3\times5+(-2)(-7) & 3\times(-2)+(-2)\times3 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 45+49 & -18-21 \\ 15+14 & -6-6 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 94 & -39 \\ 29 & -12 \end{bmatrix}$$

$\therefore B^{-1}A^{-1} = \dfrac{1}{3}\begin{bmatrix} 94 & -39 \\ 29 & -12 \end{bmatrix}$..................................(ii)

From equations (i) and (ii),

$(AB)^{-1} = B^{-1}A^{-1} = \dfrac{1}{3}\begin{bmatrix} 94 & -39 \\ 29 & -12 \end{bmatrix}$. Hence, proved.

Exercise 6.4

1. **Tick (✓) the correct option for the given questions:**

A. If the matrix A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, which of the following is the inverse matrix A^{-1}?

a. $\frac{1}{|A|}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ b. $\frac{1}{|A|}\begin{bmatrix} a & -b \\ -c & d \end{bmatrix}$ c. $\frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ d. $\frac{1}{|A|}\begin{bmatrix} -d & b \\ c & -a \end{bmatrix}$

B. In which of the following situations is the inverse matrix A^{-1} of matrix A not defined?

a. If matrix A is a singular matrix b. If matrix A is a zero matrix
c. If matrix A is a square matrix d. If matrix A is an identity matrix

C. Among the following properties, which one is the property of an inverse matrix?

a. $(A^T)^T = A$ b. $(A^{-1})^{-1} = A$
c. $(AB)^T = B^TA^T$ d. AI = IA = A

D. If the matrix A = $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, which of the following is the inverse matrix A^{-1}?

a. $\frac{1}{5}\begin{bmatrix} -2 & 3 \\ 1 & -4 \end{bmatrix}$ b. $\frac{1}{5}\begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix}$ c. $\frac{1}{5}\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$ d. $\frac{1}{5}\begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$

2. **Multiply the following matrices and prove that they are inverse matrices of each other.**

a. A = $\begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$ b. A = $\begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$ and B = $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$

c. A = $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ and B = $\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$ d. A = $\begin{bmatrix} 4 & 7 \\ 5 & 9 \end{bmatrix}$ and B = $\begin{bmatrix} 9 & -7 \\ -5 & 4 \end{bmatrix}$

3. **Find the inverse matrices of the following matrices.**

a. A = $\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$ d. B = $\begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}$ c. P = $\begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$ d. R = $\begin{bmatrix} cosA & -sinA \\ sinA & cosA \end{bmatrix}$

4. **If the following matrices are inverse of each other, find the values of *a, b* or *p, q*.**

a. $\begin{bmatrix} a & 1 \\ 5 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & -1 \\ -5 & b \end{bmatrix}$ b. $\begin{bmatrix} p & 2p-9 \\ -q & 3 \end{bmatrix}$ and $\begin{bmatrix} 3 & 5 \\ q & p \end{bmatrix}$

5. If the inverse matrix of matrix A is $A^{-1} = \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}$, find the matrix A.

6. If A = $\begin{bmatrix} 2 & 0 \\ 5 & 3 \end{bmatrix}$ and B = $\begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix}$, then,

a. Find |A| and |B| b. Find A^{-1} and B^{-1}

c. Find $(AB)^{-1}$.   d. Verify that $(AB^{-1}) = B^{-1}A^{-1}$.

Answer

1. A. c   B. a   C. b   D. d   2. Show to the teacher.

3. a. $\begin{bmatrix} \frac{4}{5} & \frac{-1}{5} \\ \frac{-3}{5} & \frac{2}{5} \end{bmatrix}$ b. $\begin{bmatrix} 1 & -2 \\ \frac{-3}{2} & \frac{7}{2} \end{bmatrix}$ c. $\begin{bmatrix} \frac{1}{2} & 0 \\ \frac{-3}{2} & 1 \end{bmatrix}$ d. $\begin{bmatrix} cosA & sinA \\ -sinA & cosA \end{bmatrix}$

4. a. $a = 3$, $b = 3$ b. $p = 2$, $q = 2$

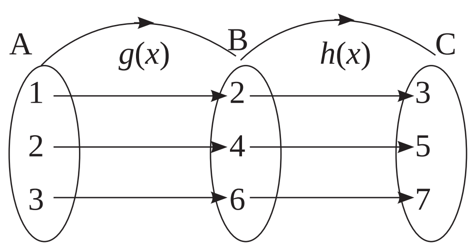
5. $\begin{bmatrix} 2 & -1 \\ -\frac{5}{2} & -\frac{1}{2} \end{bmatrix}$ 6. a. 6, 1 b. $\begin{bmatrix} \frac{1}{2} & 0 \\ \frac{-5}{6} & \frac{1}{3} \end{bmatrix}$, $\begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix}$ c. $\begin{bmatrix} \frac{11}{6} & \frac{-1}{3} \\ \frac{-41}{6} & \frac{4}{3} \end{bmatrix}$ d. Show to the teacher.

Project Work

Find the real-life applications of matrices, determinants and inverse matrices and prepare them in a newsprint with examples and present them in the class.

Miscellaneous Exercise – Within Content Area

1. In the given mapping diagram, the function g(x) is defined from set A to B, and the function $h(x)$ is defined from set B to C. Observe the two mapping diagrams and answer the following questions.



a. Write the composite function from A to C in symbolic form.

b. Write the composite function from A to C in ordered pair form.

c. Is the inverse function of the composite function from A to C defined? If defined, write it in ordered pair form.

2. If the functions are $f(x) = 2x + 3$ and $g(x) = x - 2$.

a. Find the composite functions $f(g(x))$ and $g^{-1}f(x)$.

b. When the function $f(x)$ is divided by $g(x)$, find the quotient and remainder.

c. Show the linear inequality $2x + 3y > 6$ on a graph.

d. What do you mean by surd equation? Solve $\sqrt{2x+3} = 1$ and check it.

3. The polynomials are $f(x) = x^3 - 2x^2 - 4x + 8$ and $d(x) = x + 2$

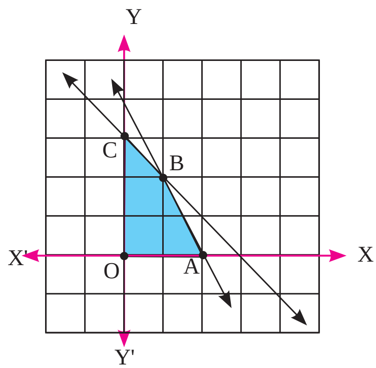
a. Write the remainder when $f(x)$ is divided by $d(x)$.

b. Is $d(x)$ a factor of $f(x)$? Give reason.

c. Find the factors of $f(x)$ using the factor theorem.

d. If $d(x)$ is a function, find $d^{-1}(x)$.

4. Observe the graph of the given system and answer to the following questions:



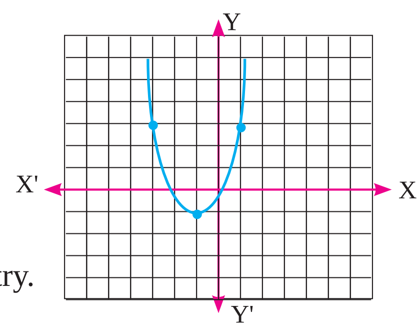
a. Write the coordinates of the points O, A, B and C.

b. From the feasible values, find the maximum and minimum value of the objective function $f(x, y) = 20x + 15y$.

c. Write the four inequalities represented by the shaded region in the graph.

d. Taking the equation of line AB as $f(x)$ and the equation of line BC as $g(x)$, find $f \circ g(x)$.

5. Observe the graph of the given system and answer to the following questions:



a. What is the name of the curve shown in the graph?

b. Write the coordinates of the vertex of the curve.

c. Find the equation of the axis of symmetry.

d. Find the equation of the curve.

6. If $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$, then,

a. State whether A^{-1} is defined or not, with reason.

b. If $A^{-1}$is defined, find A^{-1}.

c. Which of the result of A^2 among $\begin{bmatrix} 4 & 9 \\ 16 & 49 \end{bmatrix}$ and $\begin{bmatrix} 16 & -27 \\ -36 & 61 \end{bmatrix}$ find it.

7. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix}$, then,

a. What is meant by a transpose matrix? Write it.

b. Verify that: $(A + B)^T = B^T + A^T$

c. Is it possible to find AB? Give reason.

d. If AB is possible, verify that: $(AB)^{-1} = B^{-1}A^{-1}$

8. If $P = \begin{bmatrix} 2m & 7 \\ 5 & 9 \end{bmatrix}$ and the inverse matrix $Q = \begin{bmatrix} 9 & n \\ -5 & 4 \end{bmatrix}$, then,

a. Write the determinant of an identity matrix.

b. Find the values of m and n. (c) Find the determinant of P.

d. Find Q^T.

9. If $A = \begin{bmatrix} 4 & -2 \\ -3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 \\ -5 & 4 \end{bmatrix}$, then,

a. Define an inverse matrix.

b. Determine whether $(A+B)^T)^T$ equivalent to A + B? Justify the answer with calculation.

c. Find the determinant of BA.

Answer

1. a. $hg(x)$ b. f: A→C = $hg(x)$ = {(1, 3), (2, 5), (3, 7), (4, 9)}

c. The inverse of the composite function from A to C is defined because the function from A to C is one–one and onto f^{-1}: C→A = {(3, 1), (5, 2), (7, 3), (9, 4)} 2. (a) $2x - 1$, $2x + 5$ (b) 2, 7 c. and d. Show to the teacher. 3. a. 0 b. Yes, because the remainder is 0, so polynomial $d(x)$ is a factor of polynomial $f(x)$. c. $(x + 2)$, $(x - 2)$, $(x - 2)$ d. $x - 2$

4. a. O(0, 0), A(2, 0), B(1, 2), C(0, 3) b. Minimum value = 0 at O(0, 0) Maximum value = 50 at B(1, 2) c. $x \geq 0$, $y \geq 0$, $x + y \leq 3$, $2x + y \leq 4$ d. $2x - 2$

5. a. Parabola b. (–1, –1) c. $x = -1$ d. $y = (x + 1)^2 - 1$

6. a. Show to your teacher b. $\frac{1}{2}\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$ c. $A^2 = \begin{bmatrix} 16 & -27 \\ -36 & 61 \end{bmatrix}$

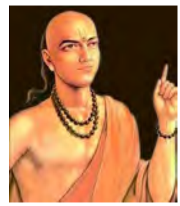
7. a. and b. Show to your teacher. c. Yes, because the number of columns of A equals the number of rows of B. d. Show to the teacher. 8. a. 1 b. m = 2, n = –7 c. 1 d. $\begin{bmatrix} 9 & -5 \\ -7 & 4 \end{bmatrix}$

9. a. Show to the teacher. b. $[(A + B)^T]^T = \begin{bmatrix} 3 & -1 \\ -8 & 6 \end{bmatrix}$ c. 2

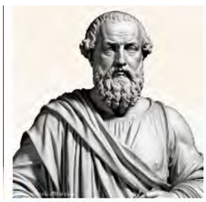
Lesson 7 Trigonometry

7.1 Introduction

The branch of mathematics that studies the relationship between the sides and angles of a triangle is called trigonometry. The development of trigonometry is believed to have started hundreds of years ago.



Aarya Bhatta



Hipparchus

Ancient Egyptians and Babylonians used the relationship between angles and sides of triangles to build pyramids. Besides this, they also used its preliminary form to measure the positions of stars and constellations. Later, the Greek mathematician Hipparchus prepared trigonometric tables using chords of circles. Therefore, he is called the father of trigonometry.

To bring trigonometry to its present form, the Indian mathematician Arya Bhatta developed the concepts of sine (jya) and cosine. After that, Islamic scholars developed the concepts of tangent and cotangent. Then, Leonhard Euler gave trigonometry the form of a function. Using it, problems ranging from daily life to science, technology, engineering, surveying, architecture, and astronomy can be solved.

7.2 Compound Angles

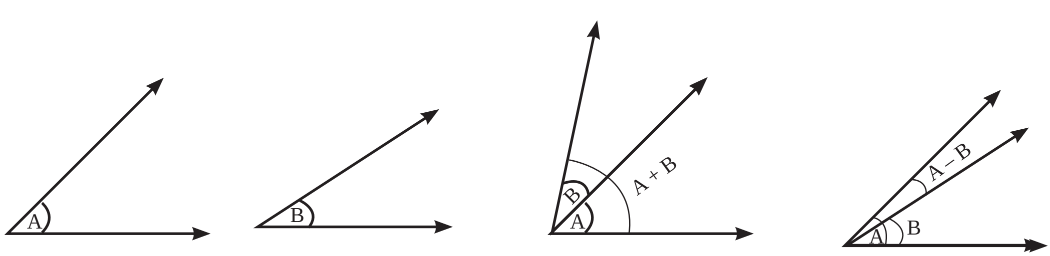
Activity 1

a. Using sticks or straws, draw angles as shown in the figure.

b. Name the larger angle as A and the smaller angle as B.

c. Now, Combine the angles to form A + B and A − B.

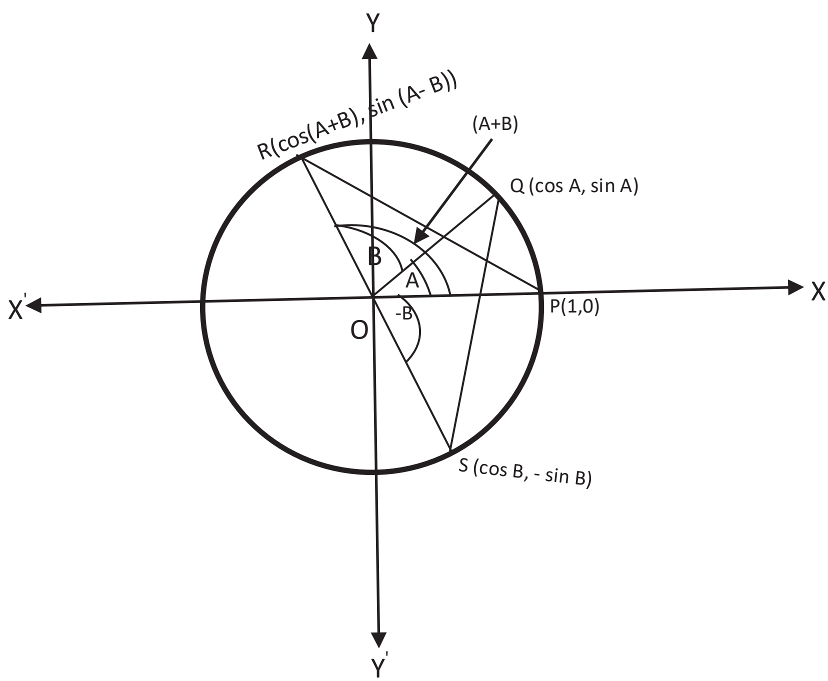
An angle formed by adding or subtracting two or more angles is called a compound angle. If A and B are two angles, A + B and A − B are compound angles.



7.2.1 Trigonometric Ratios of Compound Angles A + B and A – B

Trigonometric Ratios of cos (A + B) and cos (A – B)

A unit circle with centre O is drawn as shown in the figure. Let ∠POQ = A and ∠QOR = B. Then, ∠POR = (A + B). Construct ∠POS = B. Then, ∠QOS = (A – B). Point P lies on the X- axis (1, 0). Coordinates of points Q, R, and S can be written as Q(cos A, sin A), R(cos(A + B), sin(A + B)) and S(cos B, –sin B) respectively. Now, using the distance formula,



d $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

PR $= \sqrt{\{\cos(A+B) - 1\}^2 + \{\sin(A+B) - 0\}^2}$

$= \sqrt{\cos^2(A+B) - 2\cos(A+B) + 1 + \sin^2(A+B)}$

$= \sqrt{\cos^2(A+B) + \sin^2(A+B) - 2\cos(A+B) + 1}$

$[\because \sin^2 A + \cos^2 A = 1]$

$= \sqrt{1 - 2\cos(A+B) + 1}$

$= \sqrt{2 - 2\cos(A + B)}$

QS $= \sqrt{(\cos B - cosA)^2 + (-\sin B - sinA)^2}$

$= \sqrt{cos^2B - 2cosA\ cosB\ + cos^2A + sin^2B + 2sinA\ sinB\ + sin^2A}$

$= \sqrt{sin^2B\ + cos^2B\ + sin^2A\ + cos^2A - 2cosA\ cosB + 2sinA\ sinB}$

$= \sqrt{1 + 1 - 2cosA\ cosB + 2sinA\ sinB}$ $[\because\ sin^2A + cos^2A = 1]$

$= \sqrt{2 - 2(cosA\ cosB\ - sinA\ sinB)}$

In the circle, since chords PR and QS subtending equal central angles are equal,

PR = QS

$\sqrt{2 - 2\cos(A + B)} = \sqrt{2 - 2(cosA\ cosB\ - sinA\ sinB)}$

or, 2 – 2cos (A + B) = 2 – 2(cos A Cos B – Sin A Sin B)

So, cos (A + B) = cos A cos B – sin A sin B

and cos (A – B) = cos {A + (–B)}

= cos A cos (–B) – sin A sin (–B)

= cos A cos B + sin A sin B

Now, sin (A + B) = cos {90° – (A + B)}

= cos{(90° – A) – B)}

= cos (90° – A) cosB + sin (90° – A) sinB

= sin A cos B + cos A sin B

and sin (A – B) = sin{A + (–B)}

= sin A cos (–B) + cos A sin (–B)

= sin A cos B – cos A sin B

$$\tan(A + B) = \frac{\sin(A+B)}{\cos(A+B)}$$

$$= \frac{\sin A \cos B + \cos A \sin B}{cosA\ cosB - sinA\ sinB}$$

$$= \frac{\frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} - \frac{\sin A \sin B}{\cos A \cos B}} = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Therefore, $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

Similarly, $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

Trigonometric Ratios of Compound Angles

1. $\cos(A + B) = \cos A \cos B - \sin A \sin B$
2. $\cos(A - B) = \cos A \cos B + \sin A \sin B$
3. $\sin(A + B) = \sin A \cos B + \cos A \sin B$
4. $\sin(A - B) = \sin A \cos B - \cos A \sin B$
5. $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$
6. $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

Example 1

Prove that (without using a calculator):

$\sin 15^0 = \dfrac{\sqrt{3} - 1}{2\sqrt{2}}$

Solution: Here,

L.H.S $= \sin 15°$

$= \sin(45° - 30°)$

$= \sin 45° \cos 30° - \cos 45° \sin 30°$

$= \dfrac{1}{\sqrt{2}} \times \dfrac{\sqrt{3}}{2} - \dfrac{1}{\sqrt{2}} \times \dfrac{1}{2}$

$= \dfrac{\sqrt{3}}{2\sqrt{2}} - \dfrac{1}{2\sqrt{2}}$

$= \dfrac{\sqrt{3} - 1}{2\sqrt{2}}$ = R.H.S., proved.

Alternative Method

$\sin 15° = \sin(60° - 45°)$

$= \sin 60° \cos 45° - \cos 60° \sin 45°$

$= \dfrac{\sqrt{3}}{2} \times \dfrac{1}{\sqrt{2}} - \dfrac{1}{2} \times \dfrac{1}{\sqrt{2}}$

$= \dfrac{\sqrt{3}}{2\sqrt{2}} - \dfrac{1}{2\sqrt{2}}$

$= \dfrac{\sqrt{3} - 1}{2\sqrt{2}}$ = R.H.S., proved.

Example 2

If $\sin A = \dfrac{3}{5}$ and $\cos B = \dfrac{5}{13}$, find the values of $\sin(A + B)$ and $\tan(A - B)$.

Solution: Here,

$\sin A = \frac{3}{5}$ and $\cos B = \frac{5}{13}$

$\cos A = \sqrt{1 - \sin^2 A}$

$= \sqrt{1 - \left(\frac{3}{5}\right)^2}$

$= \sqrt{1 - \frac{9}{25}}$

$= \sqrt{\frac{25-9}{25}}$

$= \sqrt{\frac{16}{25}} = \frac{4}{5}$

$\sin B = \sqrt{1 - \cos^2 B}$

$= \sqrt{1 - \left(\frac{5}{13}\right)^2}$

$= \sqrt{1 - \frac{25}{169}}$

$= \sqrt{\frac{169-25}{169}}$

$= \sqrt{\frac{144}{169}}$

$= \frac{12}{13}$

Now, $\sin(A + B) = \sin A \cos B + \cos A \sin B$

$$= \frac{3}{5} \times \frac{5}{13} + \frac{4}{5} \times \frac{12}{13} = \frac{15}{65} + \frac{48}{65} = \frac{15 + 48}{65} = \frac{63}{65}$$

Hence, $\sin(A + B) = \frac{63}{65}$

Again, $\tan A = \frac{\sin A}{\cos A} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$

$\tan B = \frac{\sin B}{\cos B} = \frac{\frac{12}{13}}{\frac{5}{13}} = \frac{12}{5}$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} = \frac{\frac{3}{4} - \frac{12}{5}}{1 + \frac{3}{4} \times \frac{12}{5}} = \frac{\frac{15-48}{20}}{\frac{20+36}{20}} = \frac{-33}{56}$$

Hence, $\tan(A - B) = \frac{-33}{56}$

Example 3

Without using calculator prove that:

$\sin 105° - \cos 75° = \frac{1}{\sqrt{2}}$

Solution: Here,

L.H.S. = sin 105° – cos 75°

= sin(60° + 45°) – cos(45° + 30°)

= sin 60° cos 45° + cos 60° sin 45° – (cos 45° cos 30° – sin 45° sin 30°)

$= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}$

$= \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{1+1}{2\sqrt{2}} = \frac{2}{2\sqrt{2}}$

$= \frac{1}{\sqrt{2}} =$ R.H.S., proved.

Example 4

Prove that: sin(28° + α) cos(62° – α) + cos(28° + α) sin(62° – α) = 1

Solution : Here,

L.H.S. = sin(28°+ α) cos(62° – α) + cos(28° + α) sin(62° – α)

Suppose, 28° + α = A and 62° – α = B

= sin(28° + α) cos(62° – α) + cos(28° + α) sin(62° – α)

= sin A cos B + cos A sin B

= sin(A + B)

= sin(28° + α + 62° – α)

= sin 90°

= 1 = R.H.S., proved.

Example 5

Prove that: $\cot(A + B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}$

Solution : Here,

L.H.S. $\cot(A + B) = \frac{\cos(A+B)}{\sin(A+B)} = \frac{\cos \cos B - \sin A \sin B}{\sin A \cos B + \cos A \sin B}$

$$= \frac{\frac{\cos A \cos B}{\sin A \sin B} - \frac{\sin A \sin B}{\sin A \sin B}}{\frac{\sin A \cos B}{\sin A \sin B} + \frac{\cos A \sin B}{\sin A \sin B}}$$ (Dividing numerator and denominator by sin A sin B)

$= \frac{\cot A \cot B - 1}{\cot B + \cot A} =$ R.H.S., proved.

Example 6

If $A + B = \left(\frac{3\pi}{4}\right)^c$, prove that: (1 – tan A) (1 – tan B) = 2

Solution : Here,

$A + B = \left(\frac{3\pi}{4}\right)^c$

or, $\tan(A + B) = \tan\left(\frac{3\pi}{4}\right)^c$

or, $\frac{\tan A + \tan B}{1 - \tan A \tan B} = \tan 135^0$

or, $\frac{\tan A + \tan B}{1 - \tan A \tan B} = -1$

or, tan A + tan B = –1 + tan A tan B

or, 1 = – tan A – tan B + tan A tan B

or, 1 + 1 = 1 – tan A – tan B + tan A tan B [∵ adding 1 on both sides]

or, 2 = 1 (1 – tan A) – tan B (1– tan A)

or, 2 = (1 – tan A) (1 – tan B)

Therefore, (1 – tan A) (1 – tan B) = 2, proved.

Example 7

Prove that: cos (A + B) + cos(A – B) = 2 cos A cos B

Solution: Here,

L.H.S. = cos(A + B) + cos(A – B)

= cos A cos B – sin A sin B + cos A cos B + sin A sin B

= 2cos A cosB = R.H.S., proved.

Example 8

Prove that: $\frac{\cos 10^\circ + \sin 10^\circ}{\cos 10^\circ - \sin 10^\circ} = \cot 35^\circ$

Solution: Here,

R.H.S. = $\cot 35^\circ = \frac{\cos 35^\circ}{\sin 35^\circ} = \frac{\cos(45^\circ - 10^\circ)}{\sin(45^\circ - 10^\circ)}$

$= \frac{\cos 45^\circ \cos 10^\circ + \sin 45^\circ \sin 10^\circ}{\sin 45^\circ \cos 10^\circ - \cos 45^\circ \sin 10^\circ}$

$$= \frac{\frac{1}{\sqrt{2}}\cos 10^\circ + \frac{1}{\sqrt{2}}\sin 10^\circ}{\frac{1}{\sqrt{2}}\cos 10^\circ - \frac{1}{\sqrt{2}}\sin 10^\circ}$$

$$= \frac{\frac{1}{\sqrt{2}}(\cos 10^\circ + \sin 10^\circ)}{\frac{1}{\sqrt{2}}(\cos 10^\circ - \sin 10^\circ)}$$

$$= \frac{\cos 10^\circ + \sin 10^\circ}{\cos 10^\circ - \sin 10^\circ}$$

= L.H.S., proved.

Example 9

Prove that: $\tan 28° + \tan 17° + \tan 28° \tan 17° = 1$

Solution: Here,

Writing in the compound angle, $28° + 17° = 45°$

Taking tan on both sides, $\tan (28° + 17°) = \tan 45°$

or, $\frac{\tan 28^\circ + \tan 17^\circ}{1 - \tan 28^\circ . \tan 17^\circ} = 1$

or, $\tan 28° + \tan 17° = 1 - \tan 28° \tan 17°$

or, $\tan 28° + \tan 17° + \tan 28° \tan 17° = 1$

Therefore, L.H.S. = R.H.S., proved.

Exercise 7.1

1. Tick (✓) the correct option for the given questions:

A. Which one is equal to sin(A + B)?

a. sin A cos B − cos A sin B   b. sin A cos B + cos A sin B

c. cos A cos B − sin A sin B   d. sin A cos A + cos B sin B

B. Which is equal to cos(A − B)?

a. sin A cos B − cos A sin B   b. sin A cos B + cos A sin B

c. cos A cos B + sin A sin B   d. sin A cos A + cos B sin B

C. If A = 30° and B = 60°, what is the value of sin(A + B)?

a. 0   b. 1   c. $\frac{1}{2}$ d. $\frac{\sqrt{3}}{2}$

D. If $\sin(A + B) = \frac{1}{2}$ and $B = 30°$, what is the value of A?

a. 0° b. 30°

c. 45° c. 60°

E. Which of the following relations is correct?

a. $\tan(A + B) = \tan A + \tan B$ b. $\tan(A + B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

c. $\tan(A + B) = \frac{\tan A + \tan B}{1 + \tan A \tan B}$ d. $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

2. Prove that: (Without using calculator or table)

a. $\cos 15° = \frac{\sqrt{3} + 1}{2\sqrt{2}}$ b. $\sin 75° = \frac{\sqrt{3} + 1}{2\sqrt{2}}$ c. $\tan 75° = 2 + \sqrt{3}$

d. $\cos 105° = \frac{1 - \sqrt{3}}{2\sqrt{2}}$ e. $\tan 105° = -(2 + \sqrt{3})$ f. $\cot 15° = (2 + \sqrt{3})$

3. If $\sin A = \frac{3}{5}$ and $\cos B = \frac{12}{13}$, find the value of:

a. $\sin(A + B)$ b. $\cos(A + B)$ c. $\sin(A - B)$

d. $\cos(A - B)$ e. $\tan(A + B)$ f. $\cot(A - B)$

4. Prove that:

a. $\sin(A + B) + \sin(A - B) = 2\sin A \cos B$

b. $\sin(A + B) - \sin(A - B) = 2\cos A \sin B$

c. $\cos(A + B) + \cos(A - B) = 2\cos A \cos B$

d. $\cos(A + B) - \cos(A - B) = 2\sin A \cos B$

5. Prove that: (Without using calculator or table)

a. $\sin 75° - \sin 15° = \frac{1}{\sqrt{2}}$ b. $\cos 15° - \sin 75° = \frac{1}{\sqrt{2}}$

c. $\sin 105° - \cos 75° = \frac{1}{\sqrt{2}}$ d. $\cos 105° + \cos 15° = \frac{1}{\sqrt{2}}$

e. $\tan 15° + \cot 15° = 4$ f. $\cos 18° - \sin 18° = \sqrt{2} \sin 27°$

g. $\sin 35° + \cos 35° = \sqrt{2} \cos 10°$ h. $\sqrt{3} \cos 20° + \sin 20° = 2 \sin 80°$

6. **Prove that:**

a. $\sin(25° + \alpha)\cos(65° - \alpha) + \cos(25° + \alpha)\sin(65° - \alpha) = 1$

b. $\cos(25° + \alpha)\cos(65° + \alpha) - \sin(25° + \alpha)\sin(65° + \alpha) = -\sin 2\alpha$

c. $\cos(60° + \beta) - \cos\beta + \cos(60° - \beta) = 0$

d. $\dfrac{\tan(45° + \phi) - \tan\phi}{1 + \tan(45° + \phi)\tan\phi} = 1$

7. **If $A + B = \left(\frac{\pi}{4}\right)^c$, prove that:**

a. $\tan A + \tan B + \tan A \tan B = 1$

b. $(1 + \tan A)(1 + \tan B) = 2$

8. a. If $\tan X = \frac{2}{3}$ and $\tan Y = \frac{1}{5}$, prove that $X + Y = 90^0$

b. If $\tan A = m$ and $\tan B = \frac{1}{m}$, prove that $A + B = \left(\frac{\pi}{4}\right)^c$

9. **Prove that:**

a. $\dfrac{\cos 35° - \sin 35°}{\cos 35° + \sin 35°} = \tan 10°$

b. $\dfrac{\cos 17° + \sin 17°}{\cos 17° - \sin 17°} = \tan 62°$

c. $\dfrac{\cos 8° + \sin 8°}{\cos 8° - \sin 8°} = \cot 53°$

d. $\dfrac{\cos 10° - \sin 10°}{\cos 10° + \sin 10°} = \tan 35°$

10. **Prove that:**

a. $\dfrac{\tan 5A - \tan 4A}{1 + \tan 5A \tan 4A} = \tan A$

b. $\dfrac{\tan 7A - \tan 4A}{1 + \tan 7A \tan 4A} = \tan 3A$

c. $\dfrac{\tan(A+B) - \tan B}{1 + \tan(A+B)\tan B} = \tan A$

d. $\tan(A + B)\tan(A - B) = \dfrac{\tan^2 A - \tan^2 B}{1 - \tan^2 A \tan^2 B}$

e. $\dfrac{\sin(A+B)}{\sin A \sin B} = \cot B + \cot A$

f. $\dfrac{\cos(A + B)}{\sin A \sin B} = \cot A \cot B - 1$

11. **Prove that:**

a. $\tan 25° + \tan 20° + \tan 25° \tan 20° = 1$

b. $1 - \tan 27° - \tan 18° - \tan 27° \tan 18°$

c. $\tan 65° - \tan 20° = 1 + \tan 25° . \tan 20°$

d. $1 + \cot 18° + \cot 27° = \cot 18° \cot 27°$

12. Prove that:

a. $\tan 50° + \tan 60° + \tan 70° = \tan 50° \tan 60° \tan 70°$

b. $\tan 10A - \tan 6A - \tan 4A = \tan 10A \tan 6A \tan 4A$

c. $\tan 20° \tan 30° + \tan 30° \tan 40° + \tan 40° \tan 20° = 1$

d. $\cot 5A \cot 4A - \cot 9A \cot 5A - \cot 9A \cot 4A = 1$

13. Prove that:

a. $\tan 50° - \tan 40° = 2 \tan 10°$

b. $\tan 50° = 2 \tan 30° + \tan 20°$

14. Prove that:

a. $\dfrac{\sin(\beta+\gamma)}{\sin(\beta-\gamma)} = \dfrac{\tan\beta + \tan\gamma}{\tan\beta - \tan\gamma}$

b. $\dfrac{\sin(A+B).\sin(A-B).}{\cos^2 A.\cos^2 B} = \tan^2 A - \tan^2 B$

c. $\dfrac{\sin(A+B)+\sin(A-B)}{\cos(A+B)+\cos(A-B)} = \tan A$

15. Prove that:

a. $\dfrac{\sin(A-B)}{\cos A.\cos B} + \dfrac{\sin(B-C)}{\cos B.\cos C} + \dfrac{\sin(C-A)}{\cos C.\cos A} = 0$

b. $\sin(A+B+C) = \cos A \cos B \cos C (\tan A + \tan B + \tan C - \tan A \tan B \tan C)$

c. $\cos(A+B+C) = \cos A \cos B \cos C (1 - \tan A \tan B - \tan B \tan C - \tan C \tan A)$

d. $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A.\tan B.\tan C}{1 - \tan A.\tan B - \tan B.\tan C - \tan C.\tan A}$

Answer

1. A. b   B. c   C. b   D. a   E. d

3. a. $\frac{56}{65}$ b. $\frac{33}{65}$ c. $\frac{16}{65}$ d. $\frac{63}{65}$ e. $\frac{56}{33}$ f. $\frac{63}{16}$

7.3 Multiple Angles

If the value of sin 15° is given, how can we find the value of sin 30° or sin 45°?

Discuss whether we can write sin 30° = sin 2 × 15°.

Similarly, can we write sin 45° = sin 3 × 15°?

An angle formed by multiplying an angle by 2, 3, 4, ... is called a multiple angle of that angle. For example: multiple angles of A are 2A, 3A, 4A, etc.

7.3.1 Trigonometric Ratio of Multiple Angles

1. Trigonometric Ratios of Angle 2A

a. $\sin 2A = \sin(A + A)$

$= \sin A \cos A + \cos A \sin A$ [$\because \sin(A + B) = \sin A \cos B + \cos A \sin B$]

$= 2\sin A \cos A$

Therefore, $\sin 2A = 2 \sin A \cos A$

b. $\cos 2A = \cos (A + A)$

$= \cos A \cos A - \sin A \sin A$ [$\because \cos(A + B) = \cos A \cos B - \sin A \sin B$]

Therefore, $\cos 2A = \cos^2 A - \sin^2 A$

c. $\cos 2A = \cos^2 A - \sin^2 A = 1 - \sin^2 A - \sin^2 A = 1 - 2 \sin^2 A$

Therefore, $\cos 2A = 1 - 2 \sin^2 A$

d. $\cos 2A = \cos^2 A - \sin^2 A = \cos^2 A - (1 - \cos^2 A)$

$= \cos^2 A - 1 + \cos^2 A = 2 \cos^2 A - 1$

Therefore, $\cos 2A = 2 \cos^2 A - 1$

e. $\tan 2A = \tan(A + A) = \dfrac{\tan A + \tan A}{1 - \tan A \tan A} = \dfrac{2 \tan A}{1 - \tan^2 A}$

Therefore, $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

2. Transformation of sin 2A and cos 2A in terms of tan A

$\sin 2A = 2 \sin A \cos A$

$$= \frac{2 \sin A . \cos A}{1} = \frac{2 \sin A . \cos A}{\cos^2 A + \sin^2 A} \quad [\because \cos^2 A + \sin^2 A = 1]$$

$$= \frac{\frac{2 \sin A . \cos A}{\cos^2 A}}{\frac{\cos^2 A}{\cos^2 A} + \frac{\sin^2 A}{\cos^2 A}} = \frac{2 \tan A}{1 + \tan^2 A}$$

Therefore, $\sin 2A = \dfrac{2 \tan A}{1 + \tan^2 A}$

Another method

$\sin 2A = 2\sin A \cos A$

$= 2\sin A \cos A \times \frac{\cos A}{\cos A} = 2\frac{\sin A}{\cos A} \times \cos^2 A$

$= 2\tan A \times \frac{1}{\sec^2 A} = \frac{2\tan A}{1+\tan^2 A}$

Therefore, $\sin 2A = \frac{2\tan A}{1+\tan^2 A}$

$\cos 2A = \cos^2 A - \sin^2 A = \frac{\cos^2 A - \sin^2 A}{1}$

$$= \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A} = \frac{\frac{\cos^2 A}{\cos^2 A} - \frac{\sin^2 A}{\cos^2 A}}{\frac{\cos^2 A}{\cos^2 A} + \frac{\sin^2 A}{\cos^2 A}} = \frac{1-\tan^2 A}{1+\tan^2 A}$$

Therefore, $\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$

3. Trigonometric Ratios of Angle 3A

$\sin 3A = \sin(A + 2A)$

$= \sin A \cos 2A + \cos A \sin 2A$

$= \sin A(1 - 2\sin^2 A) + \cos A.\ 2\sin A . \cos A$

$= \sin A - 2\sin^3 A + 2\sin A \cos^2 A$

$= \sin A - 2\sin^3 A + 2\sin A(1 - \sin^2 A)$

$= \sin A - 2\sin^3 A + 2\sin A - 2\sin^3 A$

$= 3\sin A - 4\sin^3 A$

Therefore, $\sin 3A = 3\sin A - 4\sin^3 A$

$\cos 3A = \cos(A + 2A)$

$= \cos A \cos 2A - \sin A \sin 2A$

$= \cos A(2\cos^2 A - 1) - \sin A\ 2\sin A \cos A$

$= 2\cos^3 A - \cos A - 2\cos A \sin^2 A$

$= 2\cos^3 A - \cos A - 2\cos A(1 - \cos^2 A)$

$= 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A$

$= 4\cos^3 A - 3\cos A$

Therefore, $\cos 3A = 4\cos^3 A - 3\cos A$

Now, $\tan 3A = \tan(A + 2A) = \frac{\tan A + \tan 2A}{1 - \tan A \tan 2A} = \frac{\tan A + \frac{2\tan A}{1-\tan^2 A}}{1 - \tan A . \frac{2\tan A}{1-\tan^2 A}}$

$$= \frac{\frac{\tan A\,(1-\tan^2 A)+2\tan A}{1-\tan^2 A}}{\frac{1-\tan^2 A-2\tan^2 A}{1-\tan^2 A}}$$

$$= \frac{\tan A-\tan^3 A+2\tan A}{1-\tan^2 A} \times \frac{1-\tan^2 A}{1-3\tan^2 A} = \frac{3\tan A-\tan^3 A}{1-3\tan^2 A}$$

Therefore, $\tan 3A = \frac{3\tan A-\tan^3 A}{1-3\tan^2 A}$

Identities of Multiple angles

1. $\sin 2A = 2\sin A \cdot \cos A$
2. $\cos 2A = \cos^2 A - \sin^2 A$
3. $\cos 2A = 1 - 2\sin^2 A$
4. $\cos 2A = 2\cos^2 A - 1$
5. $\sin 2A = \frac{2\tan A}{1+\tan^2 A}$
6. $\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$
7. $\tan 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$
8. $\sin 3A = 3\sin A - 4\sin^3 A$
9. $\cos 3A = 4\cos^3 A - 3\cos A$
10. $\tan 3A = \frac{3\tan A-\tan^3 A}{1-3\tan^2 A}$

Example 1

If $\sin\theta = \frac{3}{5}$ find the values of $\sin 2\theta$, $\cos 2\theta$ and $\tan 2\theta$.

Solution: Here, $\sin\theta = \frac{3}{5}$

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{25-9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\times\frac{3}{5}\times\frac{4}{5} = \frac{24}{25}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$= \left(\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{16}{25}-\frac{9}{25} = \frac{16-9}{25} = \frac{7}{25}$$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{\frac{24}{25}}{\frac{7}{25}} = \frac{24}{25}\times\frac{25}{7} = \frac{24}{7}$$

Hence, $\sin 2\theta = \frac{24}{25}$, $\cos 2\theta = \frac{7}{25}$ and $\tan 2\theta = \frac{24}{7}$

Example 2

If **$\cos\theta = \frac{1}{2}$, find the values of $\sin 3\theta$ and $\cos 3\theta$.**

Solution

Here, $\cos\theta = \frac{1}{2}$

$$\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{4-1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

We know, $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$$= 3.\frac{\sqrt{3}}{2} - 4\left(\frac{\sqrt{3}}{2}\right)^3 = \frac{3\sqrt{3}}{2} - \frac{3\sqrt{3}}{2} = 0$$

$\cos 3\theta \quad = 4\cos^3\theta - 3\cos\theta$

$$= 4.\left(\frac{1}{2}\right)^3 - 3.\frac{1}{2} = \frac{1}{2} - \frac{3}{2} = \frac{1-3}{2} = \frac{-2}{2} = -1$$

Hence, $\sin 3\theta = 0$ and $\cos 3\theta = -1$

Example 3

Prove that: $\cot 3A = \dfrac{\cot^3 A - 3\cot A}{3\cot^2 A - 1}$

Solution

Here, L.H.S. = cot 3A

$$= \frac{1}{\tan 3A} = \frac{1}{\frac{3\tan A - \tan^3 A}{1-3\tan^2 A}} = \frac{1-3\tan^2 A}{3\tan A - \tan^3 A}$$

$$= \frac{1-3.\frac{1}{\cot^2 A}}{3.\frac{1}{\cot A} - \frac{1}{\cot^3 A}} = \frac{\frac{\cot^2 A-3}{\cot^2 A}}{\frac{3\cot^2 A-1}{\cot^3 A}} = \frac{\cot^2 A-3}{\cot^2 A} \times \frac{\cot^3 A}{3\cot^2 A-1}$$

$$= \frac{\cot A(\cot^2 A-3)}{3\cot^2 A-1} = \frac{\cot^3 A-3\cot A}{3\cot^2 A-1}$$ = R.H.S., proved.

Example 4

Prove that: $\tan A = \pm\sqrt{\dfrac{1-\cos 2A}{1+\cos 2A}}$

Solution : We know that,

$\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$

or, $\cos 2A (1 + \tan^2 A) = 1 - \tan^2 A$

or, $\cos 2A + \cos 2A \tan^2 A = 1 - \tan^2 A$

or, $\tan^2 A + \cos 2A \tan^2 A = 1 - \cos 2A$

or, $\tan^2 A (1 + \cos 2A) = 1 - \cos 2A$

or, $\tan^2 A = \frac{1 - \cos 2A}{1 + \cos 2A}$

Therefore, $\tan A = \pm \sqrt{\frac{1 - \cos 2A}{1 + \cos 2A}}$, proved.

Example 5

Prove that: $\frac{1 + \cos A + \cos 2A}{\sin A + \sin 2A} = \cot A$

Solution : Here,

$$\text{L.H.S.} = \frac{1 + \cos A + \cos 2A}{\sin A + \sin 2A} = \frac{1 + \cos A + 2\cos^2 A - 1}{\sin A + 2\sin A.\cos A}$$

$$= \frac{\cos A + 2\cos^2 A}{\sin A + 2\sin A.\cos A} = \frac{\cos A\,(1 + 2\cos A)}{\sin A\,(1 + 2\cos A)} = \frac{\cos A}{\sin A}$$

$= \cot A =$ R.H.S., proved.

Example 6

Prove that: $\frac{1 + \tan^2\left(\frac{\pi}{4} - A\right)}{1 - \tan^2\left(\frac{\pi}{4} - A\right)} = \text{cosec } 2A$

Solution : Here,

$$\text{L.H.S.} = \frac{1 + \tan^2\left(\frac{\pi}{4} - A\right)}{1 - \tan^2\left(\frac{\pi}{4} - A\right)} = \frac{1}{\frac{1 - \tan^2\left(\frac{\pi}{4} - A\right)}{1 + \tan^2\left(\frac{\pi}{4} - A\right)}} = \frac{1}{\cos 2\left(\frac{\pi}{4} - A\right)}$$

$$= \frac{1}{\cos 2\left(45^\circ - A\right)} = \frac{1}{\cos\left(90^\circ - 2A\right)} = \frac{1}{\sin 2A}$$

$= \text{cosec } 2A =$ R.H.S., proved.

Example 7

Prove that: $\cos^6\phi + \sin^6\phi = = \frac{1}{8}(5 + 3\cos 4\phi)$

Solution : Here,

L.H.S. $= \cos^6\phi + \sin^6\phi$

$= (\cos^2\phi)^3 + (\sin^2\phi)^3$

$= (\cos^2\phi + \sin^2\phi)\{(\cos^2\phi)^2 - \cos^2\phi.\sin^2\phi + (\sin^2\phi)^2\}$

$= 1\{(\cos^2\phi)^2 + (\sin^2\phi)^2 - \cos^2\phi.\sin^2\phi\}$

$= (\cos^2\phi + \sin^2\phi)^2 - 2\cos^2\phi.\sin^2\phi - \cos^2\phi.\sin^2\phi$

$= (1)^2 - 3\cos^2\phi.\sin^2\phi$

$= 1 - 3\cos^2\phi.\sin^2\phi$

$= 1 - \frac{3}{4} \times 4\cos^2\phi.\sin^2\phi$

$= 1 - \frac{3}{4}(2\sin\phi.\cos\phi)^2$

$= 1 - \frac{3}{4}(\sin 2\phi)^2$

$= 1 - \frac{3}{4}\sin^2 2\phi$

$= 1 - \frac{3}{4} \times \frac{1 - \cos 2.2\phi}{2}$ [$\because \cos 2A = 1 - 2\sin^2 A$]

$= 1 - \frac{3(1 - \cos 4\phi)}{8}$

$= \frac{8 - 3(1 - \cos 4\phi)}{8}$

$= \frac{8 - 3 + 3\cos 4\phi)}{8} = \frac{5 + 3\cos 4\phi}{8}$

$= \frac{1}{8}(5 + 3\cos 4\phi)$ = R.H.S., proved.

Exercise 7.2

1. Tick (✓) the correct option for the given questions:

A. Which of the following trigonometric expressions is equal to cos 2A?

a. $\cos^2 A + \sin^2 A$ b. $\cos^2 A - \sin^2 A$

c. $1 - 2\cos^2 A$ d. $2\sin^2 A - 1$

B. Which of the following trigonometric expressions is equal to sin 3θ?

a. $4\sin\theta - 3\sin^3\theta$ b. $4\sin^3\theta - 3\sin\theta$

c. $3\sin\theta - 4\sin^3\theta$ d. $3\sin\theta - 4\sin^4\theta$

C. If A = 30°, what is the value of sin 2A?

a. 0 b. 1 c. $\frac{1}{2}$ d. $\frac{\sqrt{3}}{2}$

D. What should be the value of A when cos 3A = –1?

a. 0° b. 30° c. 15° d. 60°

E. Which of the following trigonometric expressions is equal to tan 2A?

a. $\frac{2\tan A}{1-\tan^2 A}$ b. $\frac{2\tan A}{1+\tan^2 A}$ c. $\frac{1+\tan^2 A}{1-\tan^2 A}$ d. $\frac{1-\tan^2 A}{1+\tan^2 A}$

2. a. If sin A $= \frac{4}{5}$, find the values of sin 2A, cos 2A and tan 2A.

b. If cos A $= \frac{12}{13}$, find the values of sin 2A, cos 2A and tan 2A.

c. If cos θ $= \frac{4}{5}$, find the values of sin 3θ, cos 3θ and tan 3θ.

d. If tan $y = \frac{1}{2}$, find the value of tan 3*y*.

3. a. If cos 2A $= \frac{7}{25}$, find the value of sin A.

b. If cos 2A $= \frac{13}{36}$, find the value of cos A.

c. If tan 2A $= \frac{120}{119}$, find the value of tan A.

4. **Prove that:**

a. $\frac{1+\cos 2A}{\sin 2A} = \cot A$

b. $\frac{1-\cos 2A}{\sin 2A} = \tan A$

c. $\frac{1-\cos 2A}{1+\cos 2A} = \tan^2 A$

d. $\frac{1-\tan\theta}{1+\tan\theta} = \frac{1-\sin 2\theta}{\cos 2\theta}$

e. $\frac{\cos 2\alpha}{1+\sin 2\alpha} = \frac{1-\tan\alpha}{1+\tan\alpha}$

f. $\frac{1+\sin 2\theta-\cos 2\theta}{1+\sin 2\theta+\cos 2\theta} = \tan\theta$

g. $\frac{\sin\theta+\sin 2\theta}{1+\cos\theta+\cos 2\theta} = \tan\theta$

h. tan A + cot A = 2 cosec 2A

i. cot A – tan A = 2cot 2A

5. **Prove that:**

a. $\tan(45^0 + A) = \frac{\cos 2A}{1-\sin 2A}$

b. $\frac{1-\sin 2\theta}{\cos 2\theta} = \frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}$

c. $2\sin^2(45° - A) = 1 - \sin 2A$

d. tan(45° + A) = sec 2θ + tan 2θ

e. $2\cos^2(45° - A) = 1 + \sin 2A$

f. $\cos^2(45° - \theta) - \sin^2(45° - \theta) = \sin 2\theta$

g. $\frac{\sin 5A}{\sin A} - \frac{\cos 5A}{\cos A} = 4\cos 2A$

h. $\frac{1 - \tan^2(45° - A)}{1 + \tan^2(45° - A)} = \sin 2A$

6. a. If $\cos\theta = \frac{1}{2}(a + \frac{1}{a})$, prove that: $\cos 2\theta = \frac{1}{2}(a^2 + \frac{1}{a^2})$

b. If $\sin A = \frac{1}{2}(a + \frac{1}{a})$, prove that: $\cos 2A = -\frac{1}{2}(a^2 + \frac{1}{a^2})$

c. If $\sin\beta = \frac{1}{2}(a + \frac{1}{a})$, prove that: $\sin 3\beta = -\frac{1}{2}(a^3 + \frac{1}{a^3})$

d. If $\cos\beta = \frac{1}{2}(a + \frac{1}{a})$, prove that: $\cos 3\beta = \frac{1}{2}(a^3 + \frac{1}{a^3})$

7. **Prove that:**

a. $(\sin 2A + \sin 2B)^2 + (\cos 2A - \cos 2B)^2 = 4\sin^2(A + B)$

b. $(\sin 2A - \sin 2B)^2 + (\cos 2A + \cos 2B)^2 = 4\cos^2(A + B)$

c. $(1 + \sin 2A + \cos 2A)^2 = 4\cos^2 A(1 + \sin 2A)$

8. **Prove that:**

a. $\sin^2 A - \cos^2 A\cos 2B = \sin^2 B - \cos^2 B\cos 2A$

b. $\cos^2\alpha + \sin^2\alpha\cos 2\beta = \cos^2\beta + \sin^2\beta\cos 2\alpha$

9. **Prove that:**

a. $(2\cos\theta + 1)(2\cos\theta - 1) = 2\cos 2\theta + 1$

b. $(2\cos\theta + 1)(2\cos\theta - 1)(2\cos\theta - 1) = 2\cos 4\theta + 1$

10. **Prove that:**

a. $\frac{\cos^3 A + \sin^3 A}{\cos A + \sin A} = 1 - \frac{1}{2}\sin 2A$

b. $\cos 6\theta - \sin 6\theta = \cos 2\theta(1 - \sin^2 2\theta)$

c. $4(\sin 6\theta + \cos 6\theta) = 4 - 3\sin^2 2\theta$

d. $\cos^6\theta - \sin^6\theta = \frac{1}{4}(\cos^3 2\theta + 3\cos 2\theta)$

e. $\cos^6\theta + \sin^6\theta = \frac{1}{8}(5 + 3\cos 4\theta)$

f. $\sin^8\theta + \cos^8\theta = \frac{1}{8}(8 - 8\sin^2 2\theta + \sin^4 2\theta)$

11. **Prove that:**

a. $\text{cosec } 10° - \sqrt{3}\sec 10° = 4$

b. $\sqrt{3}\text{ cosec } 20° - \sec 20° = 4$

c. $\sec 40° + \sqrt{3}\text{ cosec } 40° = 4$

12. Prove that:

a. cosec 2A + cot 4A = cot A – cosec 4A

b. cot 8A + cosec 4A = cot 2A – cosec 8A

c. $\frac{sec4A-1}{sec2A-1}$ = tan 4A cot A

d. $\frac{sec8A-1}{sec4A-1}$ = tan8 A cot 2A

13. Prove that:

a. $\frac{1}{\tan3\theta+\tan\theta} - \frac{1}{\cot3\theta+\cot\theta} = \cot 4\theta$

b. $\frac{\cot\theta}{\cot\theta-\cot3\theta} - \frac{\tan\theta}{\tan3\theta-\tan\theta} = 1$

14. Prove that:

a. $4(\cos^3 10° + \sin^3 20°) = 3(\cos 10° + \sin 20°)$

b. $4(\cos^3 20° + \sin^3 50°) = 3(\cos 20° + \sin 50°)$

c. $\cos^3 A \cos 3A + \sin^3 A \sin 3A = \cos^3 2A$

15. Prove that:

a. $\tan \theta + \tan(60° + \theta) – \tan(60° – \theta) = 3\tan 3\theta$

b. $\cot \theta + \cot(60° + \theta) – \cot(60° – \theta) = 3\cot 3\theta$

c. $\tan \theta + 2\tan 2\theta + 4 \tan 4 \theta + 8\cot 8\theta = \cot \theta$

16. a. Express $\cos 4\theta$ in terms of $\sin \theta$.

b. Express $\cos 5\theta$ in terms of $\cos \theta$.

17. Prove that:

a. $\frac{cosA}{cosA-sinA} - \frac{cosA}{cosA+sinA} = \tan2A$

b. $\tan2\theta + \sec2\theta = \frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}$

c. $\frac{\cot2\theta+\tan2\theta}{\cot2\theta-\tan2\theta} = \sec4\theta$

d. $\frac{cos2A}{1+sin2A} = \tan(45^{\circ}-A) = \frac{1-tanA}{1+tanA}$

18. If 2tan α = 3 tan β, prove that:

a. $\tan(\alpha+\beta) = \frac{5\,sin2\beta}{5\,cos2\beta-1}$

b. $\tan(\alpha-\beta) = \frac{sin2\beta}{5-cos2\beta}$

19. If $\tan A = \frac{b}{a}$, prove that: $a \cos 2A + b \sin 2A = a$

20. Prove that: $\frac{sinA-\sqrt{1-sin2A}}{cosA-\sqrt{1-sin2A}}$ = cot A

21. Prove that: $\sin^4 A = \frac{3}{8} - \frac{1}{2}\cos 2A + \frac{1}{8}\cos 4A$

22. Prove that: $\sin^4\left(\frac{\pi}{8}\right) + \sin^4\left(\frac{3\pi}{8}\right) + \sin^4\left(\frac{5\pi}{8}\right) + \sin^4\left(\frac{7\pi}{8}\right) = \frac{3}{2}$

Answer

1. A. b B. c C. d D. d E. a

2. a. $\frac{24}{25}, \frac{-7}{25}, \frac{-24}{7}$ b. $\frac{120}{169}, \frac{119}{169}, \frac{120}{119}$ c. $\frac{117}{125}, \frac{-44}{125}, \frac{-117}{44}$ d. $\frac{11}{2}$

3. a. $\frac{3}{5}$ b. $\frac{7}{6\sqrt{2}}$ c. $\frac{5}{12}$ or $\frac{-12}{5}$

7.4 Submultiple Angles

If the value of sin 60° is given, how can we find the value of sin 30°? Can we write $\sin 15° = \sin\frac{30°}{2}$? Discuss.

Similarly, can we write $\sin\left(22\frac{1}{2}\right)° = \sin\left(\frac{45}{2}\right)°$?

An angle formed by dividing an angle with 2, 3, 4, ... equal parts is called a submultiple angle of the angle.

For example: Submultiple angles of A are $\frac{A}{2}, \frac{A}{3}, \frac{A}{4}$, etc.

7.4.1 Trigonometric Ratios of sin A, cos A and tan A in terms of Angle $\frac{A}{2}$

$\sin(A + B) = \sin A \cos B + \cos A \sin B$

$\sin A = \sin\left(\frac{A}{2} + \frac{A}{2}\right)$

$= \sin\frac{A}{2}\cos\frac{A}{2} + \cos\frac{A}{2}\sin\frac{A}{2} = 2\sin\frac{A}{2}\cos\frac{A}{2}$

Therefore, $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$

Another method

We know that, $\sin 2A = 2\sin A\cos A$

a. $\sin A = \sin 2\frac{A}{2} = 2\sin\frac{A}{2}\cos\frac{A}{2}$

Therefore, $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$

b. We know,

$\cos 2A = \cos^2 A - \sin^2 A$

$\cos A = \cos 2\frac{A}{2} = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$

Therefore, $\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$

Similarly,

c. $\cos A = 1 - 2\sin^2\frac{A}{2}$

d. $\cos A = 2\cos^2\frac{A}{2} - 1$

e. $\tan A = \dfrac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$

f. $\sin A = \dfrac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}$

g. $\cos A = \dfrac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$

7.4.2 Trigonometric Ratios of sinA, cosA and tanA in terms of angle $\frac{A}{3}$

Trigonometric ratios in terms of angle $\frac{A}{3}$

$\sin A = \sin\left(3\times\frac{A}{3}\right) = 3\sin\frac{A}{3} - 4\sin^3\frac{A}{3}$ [$\because \sin 3A = 3\sin A - 4\sin^3 A$]

Therefore, $\sin A = 3\sin\frac{A}{3} - 4\sin^3\frac{A}{3}$

Similarly, prove that:

a. $\cos A = 4\cos^3\frac{A}{3} - 3\cos\frac{A}{3}$

b. $\tan A = \dfrac{3\tan\frac{A}{3} - \tan^3\frac{A}{3}}{1-3\tan^2\frac{A}{3}}$

Identities for Multiple Angles	Identities for Submultiple Angles
1. $\sin 2A = 2\sin A\cos A$	1. $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$
2. $\cos 2A = \cos^2 A - \sin^2 A$	2. $\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$
3. $\cos 2A = 1 - 2\sin^2 A$	3. $\cos A = 1 - 2\sin^2\frac{A}{2}$

4. $\cos 2A = 2\cos^2 A - 1$	4. $\cos A = 2\cos^2\frac{A}{2} - 1$
5. $\sin 2A = \frac{2\tan A}{1+\tan^2 A}$	5. $\sin A = \frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}$
6. $\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$	6. $\cos A = \frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$
7. $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$	7. $\tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$
8. $\sin 3A = 3\sin A - 4\sin^3 A$	8. $\sin 3A = 3\sin\frac{A}{3} - 4\sin^3\frac{A}{3}$
9. $\cos 3A = 4\cos^3 A - 3\cos A$	9. $\cos A = 4\cos^3\frac{A}{3} - 3\cos\frac{A}{3}$
10. $\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}$	10. $\tan A = \frac{3\tan\frac{A}{3} - \tan^3\frac{A}{3}}{1-3\tan^2\frac{A}{3}}$

Example 1

If $\sin\frac{\theta}{2} = \frac{4}{5}$, find the values of sin θ, cos θ and tan θ

Solution: Here,

$$\sin\frac{\theta}{2} = \frac{4}{5}$$

$$\cos\frac{\theta}{2} = \sqrt{1-\sin^2\frac{\theta}{2}} = \sqrt{1-\left(\frac{4}{5}\right)^2} = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{25-16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\times\frac{4}{5}\times\frac{3}{5} = \frac{24}{25}$$

$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \left(\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right) = \frac{9}{25} - \frac{16}{25} = \frac{9-16}{25} = \frac{-7}{25}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{24}{25}}{\frac{-7}{25}} = -\frac{24}{7}$$

Therefore, $\sin\theta = \frac{24}{25}$, $\cos\theta = -\frac{7}{25}$ and $\tan\theta = -\frac{24}{7}$

Example 2

If $\sin\frac{A}{3} = \frac{1}{2}$, find the value of cos A.

Solution: Here, $\sin\frac{A}{3} = \frac{1}{2}$

We know,

$$\cos\frac{A}{3} = \sqrt{1 - \sin^2\frac{A}{3}} = \sqrt{1 - \left(\frac{1}{2}\right)^2}$$

$$= \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{4-1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

Again,

$$\cos A = 4\cos^3\frac{A}{3} - 3\cos\frac{A}{3} = 4 \times \left(\frac{\sqrt{3}}{2}\right)^3 - 3 \times \frac{\sqrt{3}}{2}$$

$$= 4 \times \frac{3\sqrt{3}}{8} - \frac{3\sqrt{3}}{2} = \frac{3\sqrt{3}}{2} - \frac{3\sqrt{3}}{2} = 0$$

Therefore, cos A = 0

Example 3

If $\sin 30° = \frac{1}{2}$, prove that : $\sin 15° = \frac{\sqrt{3} - 1}{2\sqrt{2}}$

Solution: Here,

$\sin 30° = \frac{1}{2}$

$\cos 30° = \sqrt{1 - \sin^2 30°}$

$$\cos 30° = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{4-1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

Therefore, $\cos 30° = \frac{\sqrt{3}}{2}$

or, $\cos 2 \times 15° = \frac{\sqrt{3}}{2}$

or, $1 - 2\sin^2 15° = \frac{\sqrt{3}}{2}$

or, $1 - \frac{\sqrt{3}}{2} = 2\sin^2 15°$

or, $\frac{2 - \sqrt{3}}{2} = 2\sin^2 15°$

or, $\frac{2 - \sqrt{3}}{2 \times 2} = \sin^2 15°$

or, $\frac{2 - \sqrt{3}}{2 \times 2} \times \frac{2}{2} = \sin^2 15°$

or, $\frac{4 - 2\sqrt{3}}{8} = \sin^2 15°$

or, $\frac{3 + 1 - 2\sqrt{3}}{8} = \sin^2 15°$

or, $\frac{\sqrt{3}^2 - 2 \times \sqrt{3} \times 1 + 1^2}{8} = \sin^2 15°$

or, $\frac{(\sqrt{3} - 1)^2}{(2\sqrt{2})^2} = \sin^2 15°$

or, $\sin 15° = \pm \frac{\sqrt{3} - 1}{2\sqrt{2}}$

Therefore, $\sin 15° = \frac{\sqrt{3} - 1}{2\sqrt{2}}$, proved. (Because the value of sin 15° is always positive.)

Example 4

Proved that: $\frac{1 + \cos\theta}{\sin\theta} = \cot\frac{\theta}{2}$

Solution: Here,

$$\text{L.H.S.} = \frac{1+\cos\theta}{\sin\theta} = \frac{1+2\cos^2\frac{\theta}{2}-1}{2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}} = \frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

$$= \cot\frac{\theta}{2} = \text{R.H.S., proved.}$$

Example 5

Prove that: $\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \tan\frac{\theta}{2}$

Solution: Here,

$$\text{L.H.S.} = \frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}$$

$$= \frac{1+2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}-(1-2\sin^2\frac{\theta}{2})}{1+\sin\frac{\theta}{2}.\cos\frac{\theta}{2}+2\cos^2\frac{\theta}{2}-1}$$

$$= \frac{1+2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}-1+2\sin^2\frac{\theta}{2}}{\sin\frac{\theta}{2}.\cos\frac{\theta}{2}+2\cos^2\frac{\theta}{2}}$$

$$= \frac{2\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)}{2\cos\frac{\theta}{2}\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$

$$= \tan\frac{\theta}{2} = \text{R.H.S., proved.}$$

Example 6

Prove that: $\sec A - \tan A = \tan\left(\frac{\pi}{4}-\frac{A}{2}\right)$

Solution: Here,

$$\text{R.H.S.} = \tan\left(\frac{\pi}{4}-\frac{A}{2}\right)$$

$$= \frac{\tan\frac{\pi}{4}-\tan\frac{A}{2}}{1+\tan\frac{\pi}{4}\tan\frac{A}{2}} = \frac{1-\tan\frac{A}{2}}{1+1.\tan\frac{A}{2}}$$

$$= \frac{1 - \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}}{1 + \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}} = \frac{\frac{\cos\frac{A}{2} - \sin\frac{A}{2}}{\cos\frac{A}{2}}}{\frac{\cos\frac{A}{2} + \sin\frac{A}{2}}{\cos\frac{A}{2}}} = \frac{\cos\frac{A}{2} - \sin\frac{A}{2}}{\cos\frac{A}{2} + \sin\frac{A}{2}}$$

Multiplying numerator and denominator by $\cos\frac{A}{2} - \sin\frac{A}{2}$.

$$= \frac{\cos\frac{A}{2} - \sin\frac{A}{2}}{\cos\frac{A}{2} + \sin\frac{A}{2}} \times \frac{\cos\frac{A}{2} - \sin\frac{A}{2}}{\cos\frac{A}{2} - \sin\frac{A}{2}}$$

$$= \frac{\left(\cos\frac{A}{2} - \sin\frac{A}{2}\right)^2}{\cos^2\frac{A}{2} - \sin^2\frac{A}{2}}$$

$$= \frac{\cos^2\frac{A}{2} - 2\cos\frac{A}{2}.\sin\frac{A}{2} + \sin^2\frac{A}{2}}{\cos^2\frac{A}{2} - \sin^2\frac{A}{2}}$$

$$= \frac{\cos^2\frac{A}{2} + \sin^2\frac{A}{2} - 2\sin\frac{A}{2}.\cos\frac{A}{2}}{\cos^2\frac{A}{2} - \sin^2\frac{A}{2}} = \frac{1 - \sin A}{\cos A}$$

$$= \frac{1}{\cos A} - \frac{\sin A}{\cos A}$$

= sec A – tan A = L.H.S., proved.

Exercise 7.3

1. Tick (✓) the correct option for the given questions:

A. Which of the following is equal to sin A?

a. $\cos^2\frac{A}{2} + \sin^2\frac{A}{2}$ b. $\cos^2\frac{A}{2} - \sin^2\frac{A}{2}$

c. $1 - 2\cos^2\frac{A}{2}$ d. $2\sin\frac{A}{2}\ \cos\frac{A}{2}$

B. Which of the following is equal to cos θ ?

a. $4\sin\frac{\theta}{3} - 3\sin^3\frac{\theta}{3}$ b. $4\cos^3\frac{\theta}{3} - 3\cos\frac{\theta}{3}$

c. $3\cos\frac{\theta}{3} - 4\cos^3\frac{\theta}{3}$ d. $3\sin\frac{\theta}{3} - 4\sin^4\frac{\theta}{3}$

C. If A = 60°, what is the value of $\sin\frac{A}{2}$?

a. 0 b. 1 c. $\frac{1}{2}$ d. $\frac{\sqrt{3}}{2}$

D. Which is a submultiple angle of 60°?

a. 15° b. 25° c. 35° d. 45°

E. Which is equal to tan A?

a. $\frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}$ b. $\frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$

c. $\frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$ d. $\frac{1+\tan^2\frac{A}{2}}{1-\tan^2\frac{A}{2}}$

2. a. Define submultiple angle with an example.

b. Express sin θ in terms of $\tan\frac{\theta}{2}$ c. Express cos θ in terms of $\cos\frac{\theta}{2}$

d. Express cos θ in terms of $\tan\frac{\theta}{2}$ e. Express sin θ in terms of $\sin\frac{\theta}{3}$

3. a. If $\sin\frac{A}{2} = \frac{4}{5}$, find the values of sin A, cos A and tan A.

b. If $\tan\frac{A}{2} = \frac{3}{4}$, find the values of sin A, cos A and tan A.

c. If $\cos\frac{A}{2} = \frac{5}{13}$, find the values of sin A, cos A and tan A.

d. If $\sin\frac{\theta}{3} = \frac{1}{2}$, find the values of sin θ, cos θ and tan θ.

e. If $\cos\frac{\theta}{3} = \frac{4}{5}$, find the values of sin θ, cos θ and tan θ.

f. If $\tan\frac{\theta}{3} = \frac{1}{2}$, find the value of tan θ.

4. a. If $\cos A = \frac{7}{25}$, find the value of $\sin\frac{A}{2}$.

b. If $\cos A = \frac{13}{36}$, find the value of $\cos\frac{A}{2}$

c. If $\tan A = \frac{4}{3}$, find the value of $\tan\frac{A}{2}$

5. **a. If $\cos 30° = \frac{\sqrt{3}}{2}$, prove that:**

i. $\sin 15° = \frac{\sqrt{3}-1}{2\sqrt{2}}$

ii. $\cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}}$

iii. $\tan 15° = 2 - \sqrt{3}$

iv. $\cot 15° = 2 + \sqrt{3}$

b. If $\cos 45° = \frac{1}{\sqrt{2}}$, prove that:

i. $\cos\left(22\frac{1}{2}\right)^{\circ} = \frac{1}{2}\sqrt{\left(2+\sqrt{2}\right)}$

ii. $\sin\left(22\frac{1}{2}\right)^{\circ} = \frac{1}{2}\sqrt{\left(2-\sqrt{2}\right)}$

iii. $\tan\left(22\frac{1}{2}\right)^{\circ} = \frac{1}{2}\sqrt{(3-2\sqrt{2})}$

iv. $\cot 15° = \sqrt{2} + 1$

6. **Prove that:**

a. If $\cos\frac{\theta}{2} = \frac{1}{2}\left(a + \frac{1}{a}\right)$, prove that: $\cos\theta = \frac{1}{2}\left(a^2 + \frac{1}{a^2}\right)$

b. If $\sin\frac{A}{2} = \frac{1}{2}\left(a + \frac{1}{a}\right)$, prove that: $\cos A = -\frac{1}{2}\left(a^2 + \frac{1}{a^2}\right)$

c. If $\sin\frac{\beta}{3} = \frac{1}{2}\left(a + \frac{1}{a}\right)$, prove that: $\sin\beta = -\frac{1}{2}\left(a^3 + \frac{1}{a^3}\right)$

d. If $\cos\frac{\beta}{3} = \frac{1}{2}\left(a + \frac{1}{a}\right)$, prove that: $\cos\beta = \frac{1}{2}\left(a^3 + \frac{1}{a^3}\right)$

7. **Prove that:**

a. $\frac{1+\cos A}{\sin A} = \cot\frac{A}{2}$

b. $\frac{\sin A}{1+\cos A} = \tan\frac{A}{2}$

c. $\frac{1-\cos A}{\sin A} = \tan\frac{A}{2}$

d. $\frac{\sin A}{1-\cos A} = \cot\frac{A}{2}$

e. $\frac{1+\sin A}{\cos A} = \frac{\cos\frac{A}{2}+\sin\frac{A}{2}}{\cos\frac{A}{2}-\sin\frac{A}{2}}$

f. $\cos A = \frac{1-\tan^2\frac{A}{2}}{1-+an^2\frac{A}{2}}$

g. $\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}} = \frac{1-\sin\theta}{\cos\theta}$

h. $\frac{\cos\alpha}{1+\sin\alpha} = \frac{1-\tan\frac{\alpha}{2}}{1+\tan\frac{\alpha}{2}}$

i. $\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \tan\frac{\theta}{2}$

j. $\frac{\sin\theta - \cos\frac{\theta}{2}}{1-\sin\frac{\theta}{2}-\cos\theta} = \cot\theta$

k. $\sin A + \tan A = \frac{4\tan\frac{A}{2}}{1-tan^4\frac{A}{2}}$

l. $\tan\frac{A}{2} + \cot\frac{A}{2} = 2\ \text{cosec}\ A$

8. Prove that:

a. $\tan(45° + \frac{A}{2}) = \frac{\cos A}{1-\sin A}$

b. $2\sin^2(45° - \frac{A}{2}) = 1 - \sin A$

c. $2\cos^2(45° - \frac{A}{2}) = 1 + \sin A$

d. $\cos^2(45° - \frac{\theta}{2}) - \sin^2(45^0 - \frac{\theta}{2}) = \sin\theta$

e. $\frac{1-tan^2(45° - \frac{A}{2})}{1+tan^2(45° - \frac{A}{2})} = \sin A$

f. $\frac{1-tan^2(45° - A)}{1+tan^2(45° - A)} = \sin A$

g. $\tan(45° + \frac{A}{2}) = \sec A + \tan A = \sqrt{\frac{1+\sin A}{1-\sin A}}$

h. $\frac{\sin 2A}{1+\cos 2A} \times \frac{\cos A}{1+\cos A} = \tan\frac{A}{2}$

9. **Prove that:** $\left(1+cos\frac{\pi}{8}\right)\left(1+cos\frac{3\pi}{8}\right)\left(1+cos\frac{5\pi}{8}\right)\left(1+cos\frac{7\pi}{8}\right) = \frac{1}{8}$

Answer

1. A. d B. b C. c D. a E. c

2. Show to the teacher. 3. a. $\frac{24}{25}, \frac{-7}{25}, \frac{-24}{7}$ b. $\frac{24}{25}, \frac{7}{25}, \frac{24}{7}$ c. $\frac{120}{169}, \frac{-119}{169}, \frac{-120}{119}$

d. 1, 0, ∞ e. $\frac{117}{125}, \frac{-44}{125}, \frac{-117}{44}$ f. $\frac{11}{2}$ 4. a. $\frac{3}{5}$ b. $\frac{7\sqrt{2}}{12}$ c. $\frac{1}{2}$ or -2

7.5 Transformation of Trigonometric Identities

7.5.1. Transformation of Product into Sum or Difference

In trigonometry, an expression in the form of a product of given angles can be transformed into a sum or difference. Discuss how we can change or transform a product into a sum or difference.

Compound Angle Identities

$\sin A \cos B + \cos A \sin B = \sin(A + B)$ ....(i)

$\sin A \cos B - \cos A \sin B = \sin(A - B)$ ....(ii)

Adding equations (i) and (ii),

$\sin A \cos B + \cos A \sin B + \sin A \cos B - \cos A \sin B = \sin (A + B) + \sin (A - B)$

Therefore, $2\sin A \cos B = \sin(A + B) + \sin(A - B)$.....(iii)

Subtracting equation (ii) from equation (i),

$\sin(A + B) - \sin(A - B) = \sin A \cos B + \cos A \sin B - \sin A \cos B + \cos A \sin B$

Therefore, $\sin(A + B) - \sin(A - B) = 2\cos A \sin B$.....(iv)

Similarly,

$\cos(A + B) = \cos A \cos B - \sin A \sin B$ ....(v)

$\cos(A - B) = \cos A \cos B + \sin A \sin B$..........(vi)

Adding equations (v) and (vi),

$\cos(A + B) + \cos(A - B) = \cos A \cos B - \sin A \sin B + \cos A \cos B + \sin A \sin B$

Therefore, $\cos(A + B) + \cos(A - B) = 2\cos A \cos B$.......(vii)

Subtracting equation (vi) from equation (v),

$\cos(A + B) - \cos(A - B) = \cos A \cos B - \sin A \sin B - \cos A \cos B - \sin A \sin B$

Therefore, $\cos(A + B) - \cos(A - B) = -2 \sin A \sin B$ .... (viii)

or, $\cos(A - B) - \cos(A + B) = 2\sin A \sin B$

Therefore, $2\sin A \cos B = \sin(A + B) + \sin(A - B)$

$2\cos A \sin B = \sin(A + B) - \sin(A - B)$

$2\cos A \cos B = \cos(A + B) + \cos(A - B)$

$-2\sin A \sin B = \cos(A + B) - \cos(A - B)$

$2\sin A \sin B = \cos(A - B) - \cos(A + B)$

7.5.2. Transformation of Sum or Difference into Product

From the above relations (iii), (iv), (vii) and (viii),

$\sin(A + B) + \sin(A - B) = 2\sin A \cos B$...........(i)

$\sin(A + B) - \sin(A - B) = 2\cos A \sin B$............(ii)

$\cos(A + B) + \cos(A - B) = 2\cos A \cos B$.........(iii)

$\cos(A + B) - \cos(A - B) = -2\sin A \sin B$.........(iv)

Since the sum or difference of two angles forms new angles, from the above identities,

$A + B = C$ ....(v)

$A - B = D$.............(vi)

Adding relations (v) and (vi),

$2A = C + D$

$A = \frac{C+D}{2}$

Subtracting (vi) from (v),

$2B = C - D$

$B = \frac{C-D}{2}$

Substituting the values of A, B, A + B and A – B in relations (i), (ii), (iii) and (iv),

$\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

$\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

$\cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$

Example 1

Transform the following products into sum or difference:

a. $\sin 35° \cos 25°$ b. $\cos 25° \cos 10°$ c. $\sin 10A \sin 6A$

Solution: Here,

a. $\sin 35° \cos 25° = \frac{1}{2}(2\sin 35° \cos 25°)$

$= \frac{1}{2}$ [sin (35° + 25°) + sin (35° – 25°)] $= \frac{1}{2}$ [sin 60° + sin 10°]

b. cos 25° cos 10° $= \frac{1}{2}$ (2cos 25° cos10°)

$= \frac{1}{2}$ [cos (25° + 10°) + cos (25° – 10°)] $= \frac{1}{2}$ [cos 35° + cos 15°]

c. sin 10A sin 6A $= \frac{1}{2}$ (2 sin 10A sin 6A)

$= \frac{1}{2}$ [cos(10A – 6A) – cos (10A + 6A)] $= \frac{1}{2}$ [cos 4A – cos 16A]

Example 2

Transform the following sums or differences into product form.

a. sin 55° – sin 25° b. cos 25° + cos 15° c. sin 8A – sin 6A

Solution: Here,

a. $\sin 55° - \sin 25° = 2\cos\frac{55° + 25°}{2} \cdot \sin\frac{55° - 25°}{2} = 2\cos 40° \cdot \sin 15°$

b. $\cos 25° + \cos 15° = 2\cos\frac{25° + 15°}{2} \cdot \cos\frac{25° - 15°}{2} = 2\cos 20° \cos 5°$

c. $\sin 8A - \sin 6A = 2\cos\frac{8A + 6A}{2} \sin\frac{8A - 6A}{2} = 2\cos 7A \sin A$

Example 3

Prove that: $\frac{\cos 10° + \sin 10°}{\cos 10° - \sin 10°} = \cot 35°$

Solution: Here,

L.H.S. $= \frac{\cos 10° + \sin 10°}{\cos 10° - \sin 10°} = \frac{\cos (90° - 80°) + \sin 10°}{\cos (90° - 80°) - \sin 10°}$

$= \frac{\sin 80° + \sin 10°}{\sin 80° - \sin 10°} = \frac{2\sin\frac{80° + 10°}{2} \cdot \cos\frac{80° - 10°}{2}}{2\cos\frac{80° + 10°}{2} \cdot \sin\frac{80° - 10°}{2}}$

$= \frac{2\sin 45° \cdot \cos 35°}{2\cos 45° \cdot \sin 35°} = \frac{2 \times \frac{1}{\sqrt{2}} \cos 35°}{2 \times \frac{1}{\sqrt{2}} \sin 35°} = \frac{\cos 35°}{\sin 35°}$

= cot 35° = R.H.S., proved.

Example 4

Prove that: $2\cos(45° + A)\sin(45° - A) = (\sin A - \cos A)^2$

Solution: Here,

$$\begin{aligned}
\text{L.H.S.} &= 2\cos(45° + A)\sin(45° - A)\\
&= \sin(45° + A + 45° - A) - \sin(45° + A - 45° + A)\\
&= \sin 90° - \sin 2A\\
&= 1 - \sin 2A\\
&= \sin^2 A + \cos^2 A - 2\sin A\cos A\\
&= (\sin A - \cos A)^2\\
&= \text{R.H.S., proved.}
\end{aligned}$$

Example 5

Prove that: $\sin\theta\sin(60° + \theta)\sin(60° - \theta) = \frac{1}{4}\sin 3\theta$

Solution: Here,

$$\begin{aligned}
\text{L.H.S.} &= \sin\theta\sin(60° + \theta)\sin(60° - \theta)\\
&= \frac{1}{2}\sin\theta\,[2\sin(60° + \theta)\sin(60° - \theta)]\\
&= \frac{1}{2}\sin\theta\,[\cos(60° + \theta - 60° + \theta) - \cos(60° + \theta + 60° - \theta)]\\
&= \frac{1}{2}\sin\theta\,[\cos 2\theta - \cos 120°]\\
&= \frac{1}{2}\sin\theta\,[\cos 2\theta + \frac{1}{2}]\\
&= \frac{1}{2}\sin\theta\cos 2\theta + \frac{1}{4}\sin\theta\\
&= \frac{1}{2}\times\frac{1}{2}\times 2\cos 2\theta\sin\theta + \frac{1}{4}\sin\theta\\
&= \frac{1}{4}[\sin(2\theta + \theta) - \sin(2\theta - \theta)] + \frac{1}{4}\sin\theta\\
&= \frac{1}{4}[\sin 3\theta - \sin\theta] + \frac{1}{4}\sin\theta\\
&= \frac{1}{4}\sin 3\theta - \frac{1}{4}\sin\theta + \frac{1}{4}\sin\theta
\end{aligned}$$

$= \frac{1}{4} \sin 3\theta$

= R.H.S., proved.

Example 6

Prove that: $\frac{\cos\theta - \cos 2\theta + \cos 3\theta}{\sin\theta - \sin 2\theta + \sin 3\theta} = \cot 2\theta$

Solution: Here,

$$\text{L.H.S.} = \frac{\cos\theta - \cos 2\theta + \cos 3\theta}{\sin\theta - \sin 2\theta + \sin 3\theta}$$

$$= \frac{\cos 3\theta + \cos\theta - \cos 2\theta}{\sin 3\theta + \sin\theta - \sin 2\theta}$$

$$= \frac{2\cos\frac{3\theta+\theta}{2}\cos\frac{3\theta-\theta}{2} - \cos 2\theta}{2\sin\frac{3\theta+\theta}{2}\cos\frac{3\theta-\theta}{2} - \sin 2\theta}$$

$$= \frac{2\cos 2\theta\cos\theta - \cos 2\theta}{2\sin 2\theta\cos\theta - \sin 2\theta}$$

$$= \frac{\cos 2\theta\,(2\cos\theta - 1)}{\sin 2\theta\,(2\cos\theta - 1)}$$

$$= \frac{\cos 2\theta}{\sin 2\theta} = \cot 2\theta$$

= R.H.S., proved.

Example 7

Prove that: $\sin 20° \sin 40° \sin 80° = \frac{\sqrt{3}}{8}$

Solution: Here,

L.H.S. $= \sin 20° \sin 40° \sin 80°$

$= \frac{1}{2} \sin 20° [2 \sin 80° \sin 40°]$

$= \frac{1}{2} \sin 20° [(\cos 80° – 40°) – \cos (80° + 40°]$

$= \frac{1}{2}$ sin 20° [cos 40° – cos 120°]

$= \frac{1}{2} \sin 20^\circ \, [\cos 40^\circ + \frac{1}{2}]$

$= \frac{1}{2} \sin 20^\circ \cos 40^\circ + \frac{1}{4} \sin 20^\circ$

$= \frac{1}{2} \times \frac{1}{2} \, (2\cos 40^\circ \sin 20^0) + \frac{1}{4} \sin 20^\circ$

$= \frac{1}{4} [\sin (40^\circ + 20^\circ) - \sin (40^\circ - 20^\circ)] + \frac{1}{4} \sin 20°$

$= \frac{1}{4} [\sin 60^\circ - \sin 20^\circ] + \frac{1}{4} \sin 20^\circ$

$= \frac{1}{4} [\frac{\sqrt{3}}{2} - \sin 20^\circ] + \frac{1}{4} \sin 20^\circ$

$= \frac{\sqrt{3}}{8} - \frac{1}{4} \sin 20^\circ + \frac{1}{4} \sin 20^\circ$

$= \frac{\sqrt{3}}{8}$

= R.H.S., proved.

Exercise 7.4

1. Tick (✓) the correct option for the given questions:

A. Which of the trigonometric expression is equal to 2sin Acos B?

a. sin(A + B) – sin(A – B) b. sin(A + B) + sin(A – B)

c. cos(A + B) + cos(A – B) d. cos(A + B) – sin(A – B)

B. Which of the trigonometric expression is equal to sinA – sinB ?

a. $2\sin \frac{A+B}{2} \cos \frac{A-B}{2}$ b. $-2\sin \frac{A+B}{2} \sin \frac{A-B}{2}$

c. $2\cos \frac{A+B}{2} \sin \frac{A-B}{2}$ d. $2\cos \frac{A+B}{2} \cos \frac{A-B}{2}$

C. Which of the following trigonometric expressions is equal to (cos C – cos D)?

a. $2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$ b. $2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

c. $-2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$ d. $-2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

D. What is obtained when (cos 6A + cos 4A) is transformed into product form?

a. 2sin 6A cos 4A   b. 2sin 5A cos A

d. 2cos 5A sin A   d. 2cos 5A cos A

E. sin 60° + sin 40° is equal to which of the following?

a. 2sin 50° cos 10°   b. 2sin 50° sin 10°

d. 2cos 50° sin 10°   d. 2cos 50° cos 5°

2. Transform the following trigonometric expressions in product form into sum or difference form:

a. sin 50° cos 30°   b. cos 25° cos 15°

c. cos 10A sin 6A   d. sin 27° sin 32°

3. Transform the following trigonometric expressions in sum or difference form into product form:

a. sin 50° + sin 30°   b. cos 25° + cos 15°

c. cos 7A – cos 5A   d. sin 67° – sin 43°

4. Prove that:

a. cos 50° + cos 40° = $\sqrt{2}$ cos 5°   b. sin50° – sin70° = –sin10°

c. sin 10° + sin 70° = $\sqrt{3}$ sin 40°   d. sin 10° + sin 70° = $\sqrt{3}$ sin 40°

e. cos 40° – sin 40° = $\sqrt{2}$sin 5°   f. sin 37° – cos 67° = $\sqrt{3}$ sin 7°

5. Prove that:

a. $\frac{\cos 10^\circ - \sin 10^\circ}{\cos 10^0 + \sin 10^0} = \tan 35^\circ$ b. $\frac{\cos 9^0 - \sin 9^0}{\cos 9^0 + \sin 9^0} = \cot 54^0$

c. $\frac{\cos 12^\circ - \sin 12^\circ}{\cos 12^0 + \sin 12^0} = \tan 33^\circ$ d. $\frac{\sin A - \sin B}{\sin A + \sin B} = \cot\frac{A+B}{2}\tan\frac{A-B}{2}$

e. $\frac{\cos B - \cos A}{\sin A + \sin B} = \tan\frac{A-B}{2}$ f. $\frac{\sin(60^\circ+A) + \sin(60^\circ-A)}{\cos(60^\circ+A) + \cos(60^\circ-A)} = \sqrt{3}$

g. 2sin (45° + $\emptyset$) cos (45° – $\emptyset$) = 1 + sin 2$\emptyset$

h. 2cos (45° + θ) cos (45° – θ) = cos 2θ

6. **Prove that:**

a. $\frac{\sin\theta + \sin 3\theta - \sin 2\theta}{\cos\theta - \cos 2\theta + \cos 3\theta} = \tan 2\theta$ b. $\frac{\sin 2A + \sin 3A + \sin 5A + \sin 6A}{\cos 2A + \cos 3A + \cos 5A + \sin 6A} = \tan 4A$

c. $\frac{\sin A - \sin 3A + \sin 5A - \sin 7A}{\cos A - \cos 3A - \cos 5A + \cos 7A} = \cot 3A$

d. $\frac{\cos 3A \cos 2A - \sin 5A \sin 7A}{\sin 4A \sin 3A - \sin 2A \sin 5A} = 4 \sin 5A \cos A$

7. **Prove that:**

a. $4\cos\theta \cos(60° + \theta) \cos(60° - \theta) = \cos 3\theta$

b. $4\cot\theta \cot(60° + \theta) \cot(60° - \theta) = \cot 3\theta$

c. $\sec\left(45^0 + \frac{\theta}{2}\right) \sec\left(45^0 - \frac{\theta}{2}\right) = 2 \sec\theta$

d. $\tan\left(\frac{\pi^c}{4} + \theta\right) - \tan\left(\frac{\pi^c}{4} + \theta\right) = 2 \tan 2\theta$

8. **Prove that:**

a. $\sin 10^\circ \sin 50^\circ \sin 70^\circ = \frac{1}{8}$ b. $\cos 10^\circ \cos 50^\circ \cos 70^\circ = \frac{\sqrt{3}}{8}$

c. $\sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ = \frac{1}{16}$

d. $\sin 10^\circ \sin 50^\circ \sin 60^\circ \sin 70^\circ = \frac{\sqrt{3}}{16}$

e. $\cos 20^\circ \cos 30^\circ \cos 40^\circ \cos 80^\circ = \frac{\sqrt{3}}{16}$

f. $\cos 20^\circ \cos 40^\circ \cos 80^\circ = \frac{1}{8}$

Answer

1. A. b   B. c   C. d   D. d   E. a

2. a. $\frac{\sin 80^\circ + \sin 20^\circ}{2}$ b. $\frac{\cos 40^\circ + \cos 10^\circ}{2}$ c. $\frac{\sin 16A - \sin 4A}{2}$ d. $\frac{\cos 5^\circ - \cos 59^\circ}{2}$

3. a. 2sin 40° cos 10°   b. 2cos 20° cos 5°   c. –2sin 6A sin A   d. 2cos 55° sin 12°

4 – 8. Show to the teacher.

7.6 Conditional Identities

Activity 1

Put the values of θ, A, and B as 0°, 30°, 45°, 60°, 90°, ... respectively in the following trigonometric relations. Discuss whether those relations are true for all values of the angles θ, A, and B.

a. $\sin 2\theta = 2 \sin \theta \cos \theta$

b. $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$

c. $\sin (A + B) = \sin A \cos B + \cos A \sin B$

d. $\sin A + \cos B = 1$

e. $\tan A = \cot B$

Here, relations (a), (b), and (c) are true for all angles. But relations (d) and (e) are not true for all angles.

All trigonometric relations may not be true for all angles. The trigonometric relations that are true for all angles are called trigonometric identities. For example, $\sin 2\theta = 2 \sin \theta \cos \theta$ is true for all angles, so it is an identity. But, $\tan A = \cot B$ is not true for all angles, it is only true when the condition $A + B = 90°$ is satisfied. Such relations, which are true only when certain specific conditions are satisfied, are called conditional trigonometric identities.

Example 1

Verify when $A = 0°$ and $B = 30°$: $\sin(A + B) = \sin A \cos B + \cos A \sin B$

Solution: Here,

$A = 0°$ and $B = 30°$

L.H.S. $= \sin(A + B) = \sin(0° + 30°) = \sin 30° = \frac{1}{2}$

R.H.S. $= \sin A \cos B + \cos A \sin B$

$$= \sin 0° \cos 30° + \cos 0° \sin 30° = 0 \times \frac{\sqrt{3}}{2} + 1 \times \frac{1}{2} = 0 + \frac{1}{2} = \frac{1}{2}$$

Therefore, L.H.S. = R.H.S.

Example 2

Verify when A = 0° and B = 45°: $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

Solution: Here,

L.H.S. = $\tan(A + B) = \tan(0° + 45°) = \tan 45° = 1$

R.H.S. = $\frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\tan 0° + \tan 45°}{1 - \tan 0° \tan 45°} = \frac{0 + 1}{1 - 0 \times 1} = \frac{1}{1 - 0} = 1$

Therefore, L.H.S. = R.H.S.

Example 3

Verify when the three angles of a triangle are α = 90°, β = 45° and γ = 45°:

a. $\sin(\alpha + \beta) = \sin \gamma$ b. $\cos(\alpha + \beta) = -\cos \gamma$

Solution: Here,

L.H.S. = $\sin(\alpha + \beta) = \sin(90° + 45°) = \sin 135° = \frac{1}{\sqrt{2}}$

R.H.S. = $\sin \gamma = \sin 45° = \frac{1}{\sqrt{2}}$

Therefore, L.H.S. = R.H.S., proved.

b. $\cos(\alpha + \beta) = -\cos \gamma$

Solution: Here,

L.H.S. = $\cos(\alpha + \beta) = \cos(90° + 45°) = \cos 135° = -\frac{1}{\sqrt{2}}$

R.H.S. $= -\cos\gamma = -\cos 45° = -\frac{1}{\sqrt{2}}$

Therefore, L.H.S. = R.H.S.

Example 4

When A = 30°, B = 60° and C = 90°, verify: sin 2A + sin 2B + sin 2C = 4 sin A sin B sin C

Solution: Here,

A = 30°, B = 60° and C = 90°

L.H.S. = sin 2A + sin 2B + sin 2C

= $\sin 2 \times 30° + \sin 2 \times 60° + \sin 2 \times 90° = \sin 60° + \sin 120° + \sin 180°$

$$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + 0 = \frac{\sqrt{3} + \sqrt{3}}{2} = \frac{2\sqrt{3}}{2} = \sqrt{3}$$

R.H.S. $= 4\sin A \sin B \sin C = 4\sin 30° \sin 60° \sin 90°$

$$= 4 \times \frac{1}{2} \times \frac{\sqrt{3}}{2} \times 1 = \sqrt{3}$$

Therefore, L.H.S. = R.H.S.

Example 5

In ΔABC, the three angles are $A = 30°$, $B = 120°$ and $C = 30°$. Verify that:

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$

Solution: Here,

$A = 30°$, $B = 120°$ and $C = 30°$

L.H.S. $= \tan A + \tan B + \tan C = \tan 30° + \tan 120° + \tan 30°$

$$= \frac{1}{\sqrt{3}} + (-\sqrt{3}) + \frac{1}{\sqrt{3}} = \frac{1 - 3 + 1}{\sqrt{3}} = \frac{-1}{\sqrt{3}}$$

R.H.S. $= \tan A \tan B \tan C = \tan 30° \tan 60° \tan 90°$

$$= \frac{1}{\sqrt{3}} \times (-\sqrt{3}) \times \frac{1}{\sqrt{3}} = \frac{-1}{\sqrt{3}}$$

Therefore, L.H.S. = R.H.S.

Example 6

In ΔABC, the three angles are $A = 30°$, $B = 90°$ and $C = 60°$. Verify that:

$\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C$

Solution: Here,

In ΔABC, the three angles are $A = 30°$, $B = 90°$ and $C = 60°$

L.H.S. $= \cos^2 A + \cos^2 B + \cos^2 C = \cos^2 30° + \cos^2 90° + \cos^2 60°$

$$= \left(\frac{\sqrt{3}}{2}\right)^2 + 0 + \left(\frac{1}{2}\right)^2 = \frac{3}{4} + \frac{1}{4} = \frac{3+1}{4} = \frac{4}{4} = 1$$

R.H.S. $= 1 - 2 \cos A \cos B \cos C$

$$= 1 - 2 \cos 30° \cos 90° \cos 60° = 1 - 2 \times \frac{\sqrt{3}}{2} \times 0 \times \frac{1}{2} = 1 - 0 = 1$$

Therefore, L.H.S. = R.H.S.

Exercise 7.5

1. Tick (✓) the correct option for the given questions:

A. Which of the following statements is correct?

a. Conditions are necessary in identities.

b. Conditional identities are true in all situations.

c. Identities are always true.

d. Conditional identities do not need to be verified.

B. If tan A = 1, what is the value of A?

a. 0° b. 30° c. 45° d. 60°

C. Under what condition is sin A = cos B true?

a. A = B b. A + B = 90°

c. A – B = 90° d. A + B = 180°

D. Which of the following is a conditional identity?

a. $\sin^2 A + \cos^2 A = 1$ b. $1 + \tan^2 A = \sec^2 A$

c. If A + B = 90° then, cot A = tan B d. sin 2A = 2 sin A cos A

E. If A + B + C = 180°, which one is correct?

a. sin(A + B) = cos C b. sin(A + B) = sin C

c. sin(A + B) = –sin C d. sin(A + B) = – cos C

2. The three angles of a triangle are α = 30°, β = 90° and γ = 60°. Verify that:

a. $\sin(\alpha + \beta) = \sin\gamma$ b. $\cos(\beta + \gamma) = -\cos a$

c. $\tan(\alpha + \beta) = -\tan\gamma$ d. $\cot(\alpha + \gamma) = -\cot\beta$

e. $\sin(\frac{\alpha}{2} + \frac{\beta}{2}) = \cos\frac{\gamma}{2}$ f. $\cos(\frac{\alpha}{2} + \frac{\beta}{2}) = \sin\frac{\gamma}{2}$

3. If A = 30°, B = 120° and C = 30°, verify that:

a. tan A + tan B + tan C = tan A tan B tan C

b. tan 2A + tan 2B + tan 2C = tan 2A tan 2B tan 2C

c. cot A cot B + cot B cot C + cot C cot A = 1

d. cot 2A cot 2B + cot 2B cot 2C + cot 2C cot 2A = 1

e. cot 3A cot 3B + cot 3B cot 3C + cot 3C cot 3A = 0

4. **If the three angles of ΔABC are $A = 30°$, $B = 90°$ and $C = 60°$, verify that:**
   a. $\sin 2A - \sin 2B + \sin 2C = 4 \cos A \sin B \cos C$
   b. $\sin 2A - \sin 2B - \sin 2C = 4 \sin A \cos B \cos C$
   c. $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C$
   d. $\cos 2A - \cos 2B + \cos 2C = 1 - 4\sin A \cos B \sin C$
5. **The three angles of a triangle are $\alpha = 45°$, $\beta = 45°$ and $\gamma = 90°$. Verify that:**
   a. $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 4 \sin\alpha \sin\beta \sin\gamma$
   b. $\sin^2\alpha + \sin^2\beta - \sin^2\gamma = 1 - 2\cos\alpha \cos\beta \sin\gamma$
   c. $\cos^2\alpha - \cos^2\beta + \cos^2\gamma = 1 - 2\sin\alpha \cos\beta \sin\gamma$
   d. $\cos^2\alpha - \cos^2\beta - \cos^2\gamma = 2\cos\alpha \sin\beta \sin\gamma - 1$
6. **Find out for which of the following conditions the identity Sin 2A + Sin 2B + Sin 2C = 4 Sin A Sin B Sin C is valid:**
   a. $A + B = 90°$ and $C = 90°$
   b. $A + B + C = 180°$
   c. $A + B + C = 360°$
   d. $A + B + C = 90°$

Answer

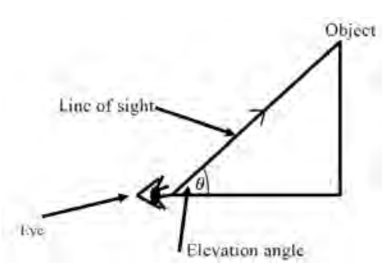
1. A. c B. c C. b D. c E. b 2 – 6. Show to the teacher.

7.7 Height and Distance

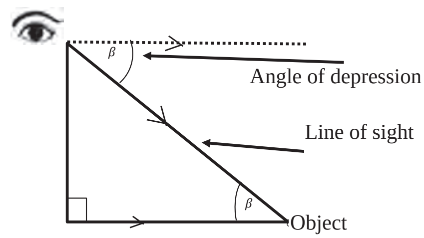
In our daily life, we often need to solve problems related to finding the height of various objects and the distance between them. For example: How tall is the school building? What is the distance between two electric poles? In all these situations, measuring the height directly is difficult. In such cases, we learn to solve problems related to height and distance using the relationship between angles and sides of a right-angled triangle and trigonometric ratios.

7.7.1 Angle of Elevation and Depression

When we look at an object above our eye level, the angle formed between the line of sight and the horizontal line is called the angle of elevation. For example: the angle formed when looking at the top of a house, pole, or tree from the ground is the angle of elevation. In the adjacent figure, θ is the angle of elevation.

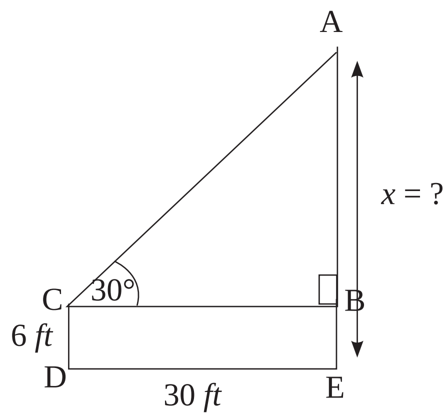


Similarly, when we look at an object below our eye level, the angle formed between the line of sight and the horizontal line is called the angle of depression. For example, while observing a bird on the yard from the roof of a house and an object on the ground observed from a tower, or a tree, the angle formed is called the angle of depression. In the given figure, the shown angle β is the angle of depression.



Example 1

A man of height 6 *ft* observes the top of a tree from a distance of 30 *ft* and finds the angle of elevation to be 30°. Find the height of the tree.



Solution: Here,

AE = height of the tree

CD = height of the man

Angle of elevation from point C to the top A of the tree, ∠ACB = 30°

Distance between the man and the tree,

DE = CB = 30 *ft*

Height of the tree (AE) = x

In right-angled triangle ABC, $\tan 30^\circ = \frac{AB}{BC}$

or, $\frac{1}{\sqrt{3}} = \frac{AB}{30}$

or, $\sqrt{3}\, AB = 30$

or, $AB = \frac{30}{\sqrt{3}} = 17.32$

AE = AB + BE

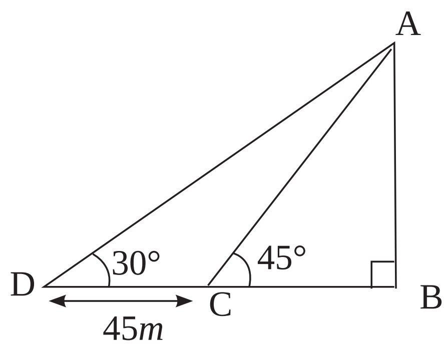
= 17.32 + 6 = 23.23

∴ Height of the tree = 23.23 *ft*

Example 2

When the angle of elevation of the top of a tree is observed from a point on the ground, it is found to be 45°. Moving 45 m away from that point, the angle of elevation of the top of the same tree becomes 30°. Find the height of the tree.

Solution: Here,



Height of the tree = AB, angles of elevation of top of the tree A formed by observing from two observation points C and D are

$\angle ACB = 45°$ and $\angle ADB = 30°$

CD = 45 *m* AB = ?

From right angled triangle ABC,

$tan\ 45^0 = \frac{AB}{BC}$

or, $1 = \frac{AB}{BC}$

or, AB = BC……(i)

Again from the right angled triangle ABD, $tan\ 30^{\circ} = \frac{AB}{BD}$

or, $\frac{1}{\sqrt{3}} = \frac{AB}{BC+CD}$

or, $\frac{1}{\sqrt{3}} = \frac{AB}{AB+45}$ [From (i)]

or, $\sqrt{3}AB = AB + 45$

or, $\sqrt{3}AB - AB = 45$

or, $AB(\sqrt{3} - 1) = 45$

or, AB (1.732 –1) = 45

or, AB (0.732) = 45

or, $AB = \frac{45}{0.732}$

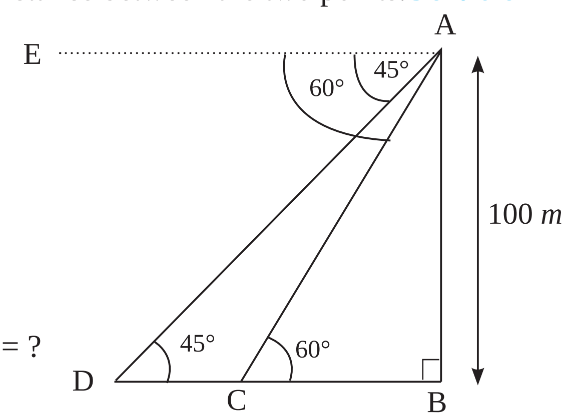
or, AB = 61.46

Therefore, Height of the tree = 61.46 m

Example 3

From the top of a cliff of height 100m, two points lying on the same side on the ground are observed. The angles of depression to these two points are found to be 60° and 45° respectively. Find the distance between the two points.**Solution:**

Here,



Height of the cliff = AB, the angle of depression from the top of the cliff at two points C and D on the ground are ∠EAC = ∠ACB = 60° and ∠EAD = ∠ADB= 45° .

AB = 100 *m*

The distance between the points (CD) = ?

From the right angled triangle ABC,

$tan\ 60^{\circ} = \frac{AB}{BC}$

or, $\sqrt{3} = \frac{AB}{BC}$

or, $\sqrt{3}\ BC = AB$

or, $\sqrt{3}\ BC = 100$

or, $BC = \frac{100}{\sqrt{3}}$

or, $BC = 57.74$ ………..(i)

Again, from right angled triangle ABD, $tan\ 45^{\circ} = \frac{AB}{BD}$

or, $1 = \frac{100}{BC + CD}$

or, $1 = \frac{100}{57.74 + CD}$ [From (i)]

or, 57.74 + CD = 100

or, CD = 100 – 57.74

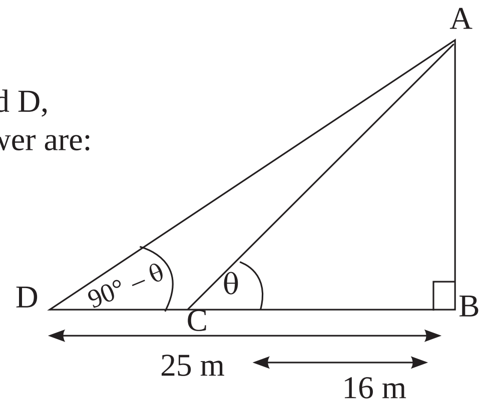
or, CD = 42.26

Therefore, distance between the points is 42.26 m.

Example 4

From two points on the same straight line and on the same side of the foot of a tower, at distances 25 m and 16 m respectively, the angles of elevation of the top of the tower are found to be complementary. Find the height of the tower.

Solution: Here,



Height of the tower = AB, From points C and D, the angles of elevation to the top A of the tower are:

$\angle ACB$ and $\angle ADB$.

BC = 16 m, BD = 25 m   AB = ?

Let $\angle ACB = \theta$, then, $\angle ADB = 90° - \theta$

Again, in right-angled triangle ABC,

$$\tan\theta = \frac{AB}{BC}$$

or, $\tan\theta = \frac{AB}{16}$ …… (i)

Again, in right-angled triangle ABD,

or, $\tan(90^\circ - \theta) = \frac{AB}{25}$

or, $\cot\theta = \frac{AB}{25}$

or, $\frac{1}{\tan\theta} = \frac{AB}{25}$

or, $\frac{\tan\theta}{1} = \frac{25}{AB}$

or, $\frac{AB}{16} = \frac{25}{AB}$ [From (i)]

or, $(AB)^2 = 16 \times 25$

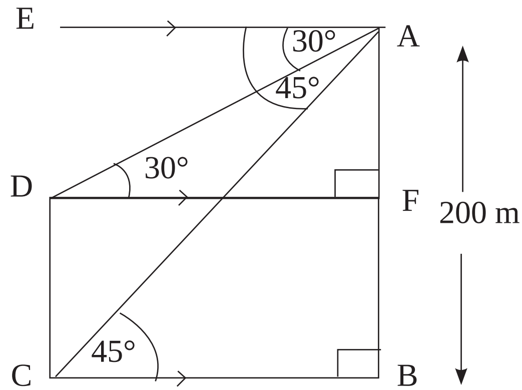
or, $(AB)^2 = 400$

or, $(AB)^2 = (20)^2$

Therefore, height of the tower = 20 m

Example 5

From the top of a hill 200 *m* high, the angles of depression of the top and the foot of a tree are found to be 30° and 45° respectively. Find the height of the tree.



Soluton: Here,

Height of the hill (AB) = 200 *m*

From the top A of the hill, the angles of depression to the foot C and the top D of the tree are:

∠EAC = ∠ACB = 45° and

∠EAD = ∠ADF = 30°

Height of the tree (CD) = ?

Again, in right-angled triangle ABC, $\tan 45^\circ = \frac{AB}{BC}$

or, $1 = \frac{AB}{BC}$

or, $1 = \frac{200}{BC}$

or, BC = 200 ……….(i)

In right-angled triangle AFD, $\tan 30^0 = \frac{AF}{DF}$

or, $\frac{1}{\sqrt{3}} = \frac{AF}{BC}$

or, $\frac{1}{\sqrt{3}} = \frac{AF}{200}$ [∵ from (i)]

or, $\sqrt{3}$ AF = 200

or, $AF = \frac{200}{\sqrt{3}}$

or, AF = 115.47

Now, CD = BF = AB – AF

or, CD = 200 – 115.47

or, CD = 84.53

∴ Height of the tree = 84.53 m

Exercise 7.6

1. **Tick (✓) the correct option for the given questions:**

   A. What is the angle called which is formed by the line of sight with the horizontal when looking at an object below the eye level?

   a. Angle of depression　　b. Acute angle

   c. Angle of elevation　　d. Right angle

   B. What is the angle called that is formed by the line of sight with the horizontal when looking at an object above the eye level?

   a. Angle of depression　　b. Acute angle

   c. Angle of elevation　　d. Right angle

   C. If the angle of elevation of an object observed from a place is 0°, where is that object?

   a. Above　　b. Below

   c. At eye level　　d. Not visible

   D. When the angle of elevation of the top of a pole observed from a place is 45°, what is the relationship between the height of the pole and the distance from that place to the pole?

   a. Height > Distance　　b. Height < Distance

   c. Equal　　d. No relation

   E. As the angle of elevation of the top of a tower observed from a place increases, what happens to the distance?

   a. Increases　　b. Decreases

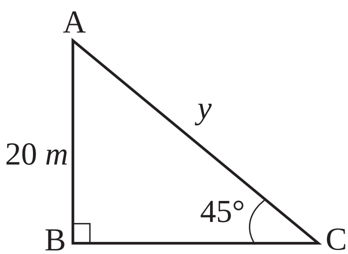
   c. Remains the same　　d. No relation

   F. What is the angle of elevation when observing the top of a tree having the height of $10\sqrt{3}$ m from a point 10 m away?

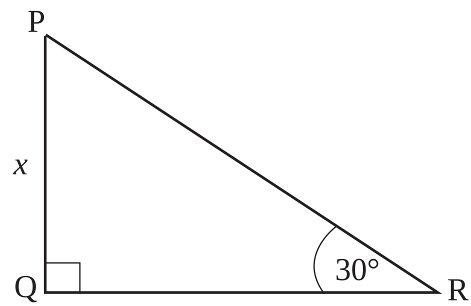
   a. 30°　　b. 45°　　c. 60°　　d. 120°

2. **Find the values of x, y, and θ in the given figures:**

a.



b.



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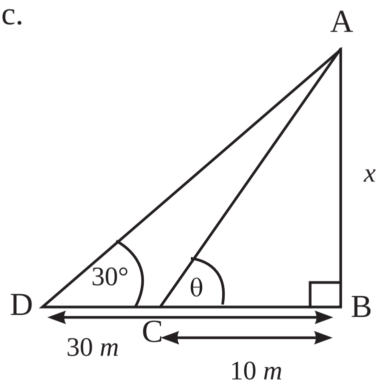

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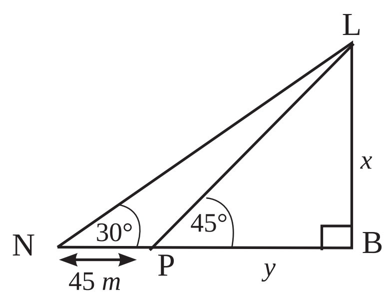
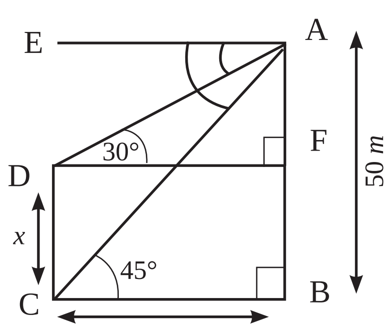
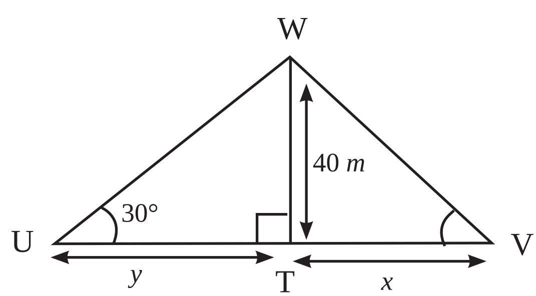


e.



f.



3. a. When observing a point on the roof of a house from a point on the ground, the angle of elevation was found to be 60°. When moving 50 m away from the house and observing again, the angle of elevation was found to be 30°. Find the height of the house.

   b. The angles of elevation of the top of 30 ft tall pole when observed from two points on the same side are found to be 60° and 45° respectively. Find the distance between those two points.

4. a. From the top of a hill of 500 m high, two points on the same side of the ground are observed. The angles of depression found are 60° and 30° respectively. Find the distance between the two points.

   b. From the roof of a house, two people sitting on the ground at a distance of 10 ft are observed. The angles of depression found are 60° and 45° respectively. Find the height of the house.

5. a. The angles of elevation of top of the pole are found to be supplementary when it is observed from two points 40 m and 36 m away from the base of the pole. Find the height of the pole.

   b. The heights of two poles 50 m apart, and the height of one pole is double of other. From a point midway between the poles, the angles of elevation of their tops are found to be supplementary. Find the heights of the poles.

6. a. The shadow of a vertical pole on horizontal ground is 50 m longer when the sun's altitude is 30° than when it is 60°. Find the height of the pole.

   b. When the sun's altitude is 30°, the shadow of a pole of 35 ft high is x m longer then the shadow when the sun altitude is 45°. Find the value of x.

7. a. From the roof of a 30 m tall building, the angles of depression of the top and bottom of a column are found to be 45° and 60° respectively. Find the height of the column.

   b. The pinnacle of a temple is observed from the roof and the foot of a building. The angles found are depression 60° and elevation 30° respectively. If the height of the building is 20 *m*, find the height of the temple.

8. a. On top of a tower, a flag of Nepal is placed. From a point 100 *m* away from the foot of the tower, the angles of elevation of the top of the tower and the top of the flag are found to be 45° and 60° respectively. Find the height of the flag.

   b. A man of height 1.8 m observes the roof of a house and the top of a wall. The angles of elevation found are 45° and 30° respectively. If the top of the wall is at a height of 20 *m* from the ground, find the height of the house.

9. a. From the foot and the roof of a 15 *m* tall house, the angles of elevation of the top of a column are found to be 60° and 45° respectively. Find the height of the column.

b. An airplane flying horizontally at a constant height of 6000 m from the ground is observed from a point on the ground at an angle of elevation of 30°. After 10 seconds, it is observed from the same point at an angle of elevation of 45°. Find the speed of the airplane in km/hr.

10. a. There are two poles of equal height on opposite sides of a 30 m wide road. From a point between the two poles, the angles of elevation of the tops of the poles are found to be 60° and 30° respectively. Find the height of the poles and the distances from the observation point to the poles.

b. From a point between two poles, a man of height 1.6 m observes the tops of both poles. The angles of elevation found are 60° and 45° respectively. If the height of the shorter pole is 41.6 *m*, find the height of the taller pole and the distance between the poles.

11. a. A pole is divided by a point from the ground in the ratio 4:5. If both the parts subtend equal angles at a point 20 m from the base, find the height of the pole.

b. On the top of a tower, a flag of height 5 m is fixed. From a point on the ground, the tower and the flag subtend angles of 45° and 15° respectively. Find the height of the tower.

12. Two poles of equal height are given. Standing at a point between the two poles, a person observes the top of each pole and finds the angle of elevation to be 45°. Moving 20 m closer to one pole, the angle of elevation becomes 60°.

a. Represent the given context in a diagram.

b. Find the height of the poles.

c. Find the distance between the two poles.

d. As the observer moves closer to the pole, the angle of elevation increases. Explain with reason.

13. Outside a building, a ladder of length 10 m is placed such that its top rests at a point 4 m below the roof of the building. From the foot of the ladder, the angle of elevation to the roof of the building is 60°.

a. Draw a diagram representing the given context.

b. Find the height on the building where the top of the ladder touches.

c. Find the distance between the foot of the ladder and the building.

d. Find the angle of elevation of a point that is 20 m below the roof from the foot of the ladder.

Answer

1. A. d B. a C. c D. c E. c F. b

2. a. $y = 20\sqrt{2}$ m b. $x = 5.77$ m c. $x = 10\sqrt{3}$ m, $\theta = 60^0$

d. $x = 61.48$ m, $y = 61.48$ m e. $x = 21.13$ m, $y = 50$ m f. $x = 40$, $y = 40\sqrt{3}$

3. a. 43.30 m b. 12.68 ft. 4. a. 577.35 m b. $x = 23.66$ ft. 5. a. 37.95 m

b. 17.68 m, 35.36 m 6. a. 43.30 m b. 25.62 m 7. a. 12.68 m b. 5 m

8. a. 73.2 m b. 33.32 m 9. a. 35.49 m b. 1581.23 km/h 10. a. 12.99 m

b.15 m 11. a. 15 m b. 6.83 m 12. b. 47.32 m c. 94.64 m

13. b. 7.49 m c. 6.63 m d. 48.57°

Project Work

Using the clinometer, from a point within the school premises, measure the angle of elevation of the top of a nearby tree or the roof of the school building. Also measure the distance from the observation point to the tree or the building, and find its height.

From different points lying on the same straight line, measure the angles and distances, and investigate the relationship between them. Prepare a report and present it in the class.

Miscellaneous Exercise - Within content area

1. Given: $\sqrt{2}$ cos A sin B = $\frac{1}{2}$ and tan A + cot B = 2.

   a. Write any two compound angles of A and B.

   b. Find the value of (A – B).

   c. Is the relation cos (A – B) = cos A cos B + sin A sin B valid for all values of A and B? Write with reason.

2. Given: cos 45° = $\frac{1}{\sqrt{2}}$

   a. Express cos A in terms of $\sin\frac{A}{2}$

   b. Write any two submultiple angles of 15°.

   c. Prove that: $\sin\left(22\frac{1}{2}\right)° = \frac{1}{2}\left(\sqrt{2-\sqrt{2}}\right)$

3. The three angles of triangle ABC are A = 120°, B = 30° and C = 30°.

a. Write cos 2A in terms of cos A.

b. Verify using the given angles: tan A + tan B + tan C = tan A tan B tan C

c. Is tan A = cot B true for the given triangle ABC? Write with reason.

4. A 9 m long ladder is placed against the outer wall of a house, 9 m below the roof, such that it makes an angle of 60° with the ground at a certain point.

   a. Represent the given context in a diagram.

   b. Find at what height on the house the top of the ladder touches.

   c. Find how far the foot of the ladder is from the house.

   d. Find at how many meters below or above the roof, the top of the ladder makes a 30° angle of elevation.

5. There are two poles of equal height. A person standing midway between both poles observes the top of each pole at an angle of elevation of 60°. When moving 60 m closer to one pole, he observes the top of the opposite pole at an angle of elevation of 45°.

   a. Define angle of elevation.

   b. Represent the given context in a diagram.

   c. Find the height of the poles.

   d. What is the relationship between the distance between the man and the pole and the angle of elevation? Write it.

Answer

1. a. A + B and A – B   b. 45°   c. Valid because it is an identity.

2. a. $\cos A = 1 - 2\sin^2 \frac{A}{2}$ b. 30°, 45°

3. a. $2\cos^2 A - 1$ 4. b. 7.79 m   c. 4.5 m   d. 12.29 m below

5. c. 141.97 m   d. Inverse variation

Lesson 8

Coordinate Geometry

8.0 Introduction

Coordinate geometry is algebraic geometry that expresses the location of any point numerically using coordinates (x, y). French mathematician René Descartes conceptualized coordinate geometry. The Cartesian coordinate system is named after him. Pierre de Fermat and Apollonius of Perga have also contributed to this field. Their work involved expressing points as ordered pairs of numbers (x, y), and representing straight lines, curves, and geometric figures through equations. Furthermore, they worked on expressing circle, parabola, ellipse, hyperbola etc., in an algebraic form. The Cartesian coordinate system, by providing a common language for algebra and geometry, is found to be used in physics, computer science, engineering, and astronomy.

René Descartes

8.1 Equation of Lines Passing through Two Points

a. Equation of a straight line in point slope form

If the slope (m) of a line and a point (x_1, y_1) on the line are known, the equation of the line can be obtained as:

In the figure, slope of line AB is $\tan\theta = m$. Let us take a point $P(x, y)$ on the line AB. Here, the equation of line with slope (m) and y-intercept (c) is given by

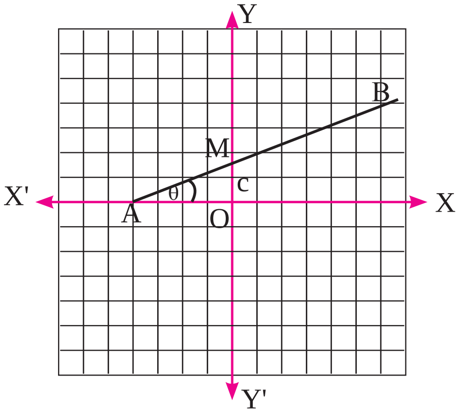
$y = mx + c$ ……………………………(i)

The given line passes through $Q(x_1, y_1)$, therefore the point satisfies the given line (i).

So, $y_1 = mx_1 + c$ ………………….(ii)

Now, subtracting equation (ii) from the equation (i),

$$y - y_1 = mx + c - (mx_1 + c)$$



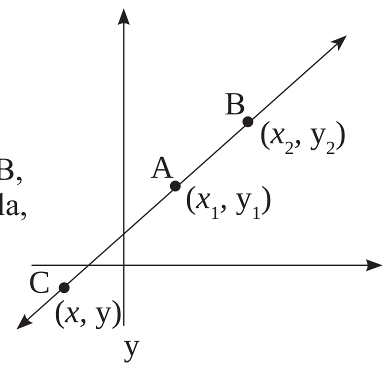
or, $y - y_1 = mx + c - mx_1 - c$

or, $y - y_1 = mx - mx_1$

or, $y - y_1 = m(x - x_1)$

Therefore, the equation of line passing through $P(x_1, y_1)$ and with slope (m) is $y - y_1 = m(x - x_1)$.

b. Equation of Lines Passing through Two Points



Let, there are two points $A(x_1, y_1)$ and $B(x_2, y_2)$.

There is a line AB passing through the points A and B, and a point $C(x, y)$ lies on the line AB. By the formula,

slope of line AB = $(m_1) = \frac{y_2 - y_1}{x_2 - x_1}$

If the points A, B and C lie on the same line, their slopes are equal.

Therefore, slope of line AB (m_1) = slope of line CA (m_2)

Let, $A(x_1, y_1,) = (x_1, y_1)$ and $C(x, y) = (x_2, y_2,)$

By the formula, $(m_2) = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_1}{x - x_1}$

If the points C, A and B lie on the same line, the slopes of AB and CA will be equal.

Therefore, slope of line segment AB (m_1) = slope of line segment CA (m_2).

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

or, $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

Therefore, the equation of line passing through two given points (x_1, y_1) and (x_2, y_2) is

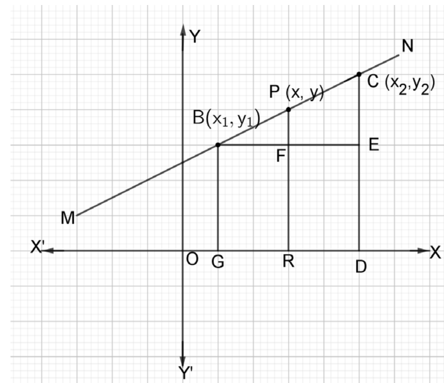
$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

Thought Provoking Question: Will we obtain the same equation by equating the slopes of AB and BC? Think and discuss it.

Another method

In the figure alongside, a straight line MN passes through $B(x_1, y_1)$ and $C(x_2, y_2)$. Let us take a point $P(x, y)$ on it.

Now, let us draw perpendiculars BG, PR and CD to OX. Similarly, let us draw perpendicular BE to CD. It intersects PR at F. Here,



BF = GR = OR – OG = $x - x_1$

BE = GD = OD – OG = $x_2 - x_1$

Similarly, PF = PR – FR = PR – BG

$= y - y_1$

CE = CD – ED = CD – BG $= y_2 - y_1$

Triangles BPF and BCE are similar triangles. (How?)

Therefore, $\frac{BF}{BE} = \frac{PF}{CE}$

or, $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} =$ where, $x_1 \neq x_2$ and $y_1 \neq y_2$

or, $y - y_1 = \frac{x - x_1}{x_2 - x_1}(y_2 - y_1)$

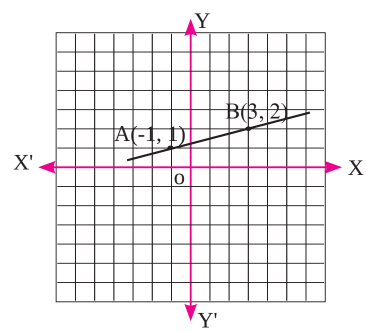
or, $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

Therefore, the equation of line passing through two given points (x_1, y_1) and (x_2, y_2) is

$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

Example 1

Find the equation of a straight line passes through the points A(–1, 1) and B(3, 2) given in the figure.



Solution

Let, A(–1, 1) = (x_1, y_1) and B(3, 2) = (x_2, y_2)

By the formula, equation of straight line passing through two points is $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

or, $y - 1 = \frac{2 - 1}{3 - (-1)}\{x - (-1)\}$

or, $y - 1 = \frac{1}{3 + 1}(x + 1)$

or, $y - 1 = \frac{1}{4}(x + 1)$

or, $4(y - 1) = x + 1$

or, $4y - 4 = x + 1$

Therefore, $x - 4y + 5 = 0$ is required equation of the line AB.

Thought Provoking Question: While finding the equation of a line passing through two points, does it make any difference which point is taken as (x_1, y_1) and which as (x_2, y_2)? Justify your answer with an example.

Example 2

The points A(−5, 3), B(−10, 6) and C(5, −3) are given.

a. Find the equation of the line AB.

b. Prove that the given three points are collinear.

Solution : Here,

Let, A(–5, 3), = (x_1, y_1) and B(–10, 6)) = (x_2, y_2)

a. Using the formula for the equation of a line passing through two points:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

or, $y - 3 = \frac{6-3}{-10-(-5)}\{x - (-5)\}$

or, $y - 3 = \frac{3}{-10+5}(x + 5)$

or, $y - 3 = \frac{3}{-5}(x + 5)$

or, $-5(y - 3) = 3x + 15$

or, $0 = 3x + 15 + 5y - 15$

Therefore, $3x + 5y = 0$……...(i) is required equation of the given line.

Thought provoking question: If A and C or B and C are taken instead of A and B, will equation of the line be same or not?

b. Now substituting the point C(5,–3) in the equation (i),

$3 \times 5 + 5 \times (-3) = 0$

or, $15 - 15 = 0$

or, $0 = 0$ which is true.

So, third point C(5, –3) also lies on the line represented by the equation (i) .

Therefore, A(−5, 3), B(−10, 6) and C(5, −3) are collinear points. Hence, the equation of line passing through A(−5, 3), B(−10, 6) and C(5, −3) is $3x + 5y = 0$.

Example 3

There is a line that passes through intersecting points of two straight lines $3x + y = 2$ and $5x + 2y = 3$ and the point (3, 1).

a. Find the coordinates of the point of intersection of the two straight lines.

b. Find the equation of the straight line passing through the point of intersection and the point (3, 1).

Solution: Here,

a. The equations of the two straight lines are:

$3x + y = 2$ ....(i) and $5x + 2y = 3$..................(ii)

From equation (i), $y = 2 - 3x$

Substituting the value of y in equation (ii):

or, $5x + 2(2 - 3x) = 3$ or, $5x + 4 - 6x = 3$

or, $-x = 3 - 4$ or, $x = 1$

Again, substituting $x = 1$ in $y = 2 - 3x$

$y = 2 - 3 \times 1 = -1$

Therefore, the point of intersection of the two lines is (1,–1).

b. Let, $(3, 1) = (x_1, y_1)$ and $(1, -1) = (x_2, y_2)$

By the formula, the equation of a line passing through two points:

$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

or, $y - 1 = \frac{-1 - 1}{1 - 3}(x - 3)$

or, $y - 1 = \frac{-2}{-2}(x - 3)$

or, $y - 1 = (x - 3)$

or, $0 = x - y - 3 + 1$ or, $x - y - 2 = 0$

Therefore, the required equation is $x - y - 2 = 0$

Exercise 8.1

1. **Tick (✓) the correct option for the given questions:**

A. Which one is the equation of straight line in two points form?

a. $y = mx + c$ b. $\frac{x}{a} + \frac{y}{b} = 1$

c. $y - y_1 = m(x - x_1)$ d. $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

B. Which one is the equation of line passing through the points (2, 3) and (–1, 0) from given below?

a. $x - y + 1 = 0$ b. $x + y + 1 = 0$

c. $x - y - 1 = 0$ d. $-x - y - 1 = 0$

C. Which is the equation of line passing through $(a, 0)$ and $(0, b)$?

a. $\frac{x}{b} + \frac{y}{a} = 1$ b. $\frac{x}{a} + \frac{y}{b} = 1$

c. $bx + ay = 0$ d. $ax + by = 0$

D. Which one of the following represents the equation of line passing through the points (–4, –2) and (–3, 5) from given below?

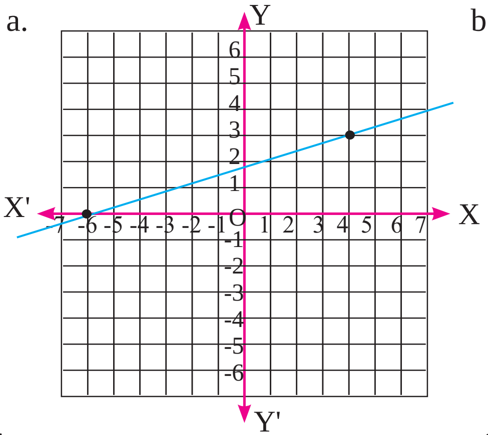
a. $y = 7x + 26$ b. $y = -7x + 26$

c. $y = -7x - 26$ d. $y = \frac{1}{7}x + 26$

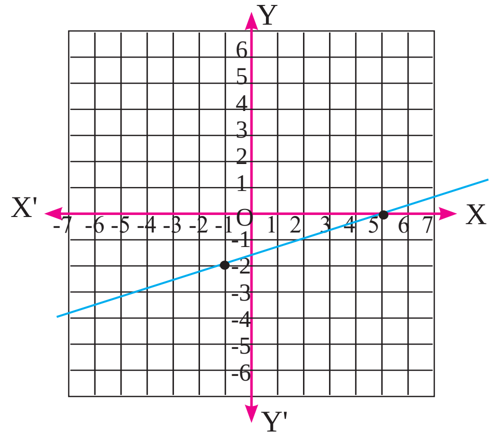
2. How can the equation of a straight line be determined when two points are given? Explain.

3. Find the equation represented by the given graph:

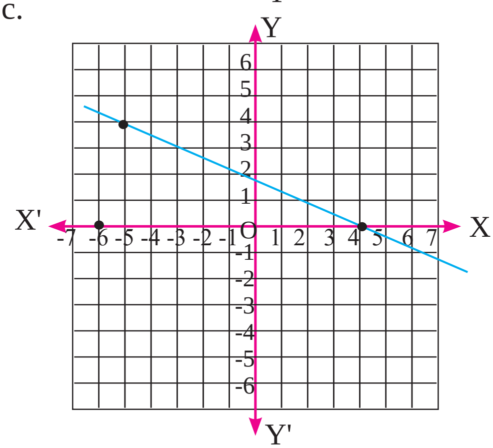
a.



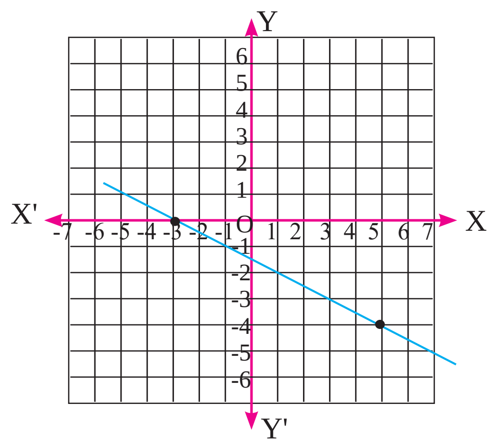
b.



c.



d.



4. Find the equation of the straight line passing through the given two points:

a. (2, 3) and (–1, 0) b. (4, 3) and (–2, 4) c. (6, –3) and (–1, –5)

d. (0, $-a$) and (b, 0) e. (p, 0) and (0, q) f. (4, 3) and (5, 6)

5. Prove that the following three points are collinear:

a. (1, –1), (5, 2) and (9, 5) b. (–3, 0), (0, –6) and (–1, –4)

c. (2, 4), (4, 6) and (6, 8) d. (1, –1), (2, 1) and (4, 5)

e. (7, –2), (2, 3) and (–1, 6)   f. (0, 4), (3, –2) and (6, 0)

6. **If three points A(b, 0), B(0, c) and C(3, 3) lie on the same straight line:**

   a. Can the point B(0, c) be called the midpoint of line segment AC? Give reason.

   b. Using the equation of a line through two points, prove that: $\frac{1}{b} + \frac{1}{c} = \frac{1}{3}$

7. **If three points A(p, 0), B(0, q) and C(5, 5) lie on the same straight line:**

   a. What is meant by saying that the given points are collinear? Write.

   b. Using the equation of a line through two points, prove that: $\frac{1}{p} + \frac{1}{q} = \frac{1}{5}$

8. **The points P(1, 0), Q(0, 1) and R(2, 3) are the vertices of triangle PQR:**

   a. Draw triangle PQR with the given points.

   b. Find the midpoint of side QR.

   c. Find the equation of the line (median) joining vertex P to the midpoint of QR

9. **The points A(2, 2), B(2, 8) and C(−6, 2) are the vertices of triangle ABC:**

   a. Find the midpoint of side BC.

   b. Find the length of median AM.

   c. Prove that the equation of the median from A(2, 2) is $3x + 4y = 14$.

10. **The points X(−1, 3), Y(1, –1) and Z(5, 1) are the vertices of triangle XYZ:**

    a. Find the length of median XD drawn from X(−1, 3).

    b. Find the equation of median XD.

11. **Two points A(a, b) and B(b, a) lie on the lines $6x – y = 1$ and $2x – 5y = 5$ respectively.**

    a. Find the values of a and b.   b. Write the coordinates of points A and B.

    c. Find the distance between A and B.   d. Find the equation of line AB.

Answer

1. A. d   B. a   C. b   D. a   2. Show to the teacher.
3. a. $x – 3y + 5 = 0$ b. $x – 3y – 5 = 0$ c. $2x + 3y – 15 = 0$ d. $x + 2y + 4 = 0$
4. a. $x – y + 1 = 0$ b. $x + 6y – 22 = 0$ c. $2x – 7y – 33 = 0$
d. $ax – by – ab = 0$ e. $qx + py – pq = 0$
f. $3x – y – 9 = 0$ 5 - 7. Show to the teacher.   8. a. Show to the teacher.   b. (1, 2)
c. $x = 1$ 9. a. (–2, 5)   b. 5 units   10. a. 5 units   b. $3x + 4y – 9 = 0$
11. a. $a = 1, b = 5$ b. A (1, 5) and B (5, 1)   c. $4\sqrt{2}$ units   d. $x + y – 6 = 0$

Project Work

Select any two locations within the school premises, such as:

a. Main gate b. Playground c. Classroom d. Library, etc.

Represent these two locations on a graph paper or Cartesian plane as two points with coordinates: (x_1, y_1) and (x_2, y_2).

a. Calculate the slope of the line joining these two points.

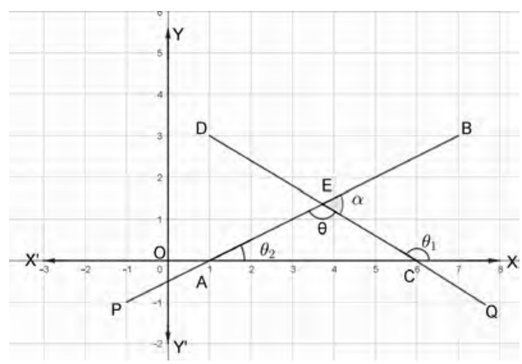
b. Find the equation of the line joining the two points.

Explain how the obtained equation of the line represents the straight path between those two locations in the school.

c. If the path is shifted 5 meters to the right or left, what will be the new equation of the line? Explain. Also, discuss with your teacher whether the slopes of the two equations will be the same or different, and draw a conclusion.

8.2 Angle between Two Straight Lines

In the given figure, the straight lines PB and DQ intersect at point E, forming angles $\angle QEP = \theta$ and $\angle QEB = \alpha$. In the slope-intercept form, the equations of lines DQ and PB are respectively $y = m_1x + c$ and $y = m_2x + c$. The angles made by lines DQ and PB with the X- axis are respectively $\angle DCX = \theta_1$ and $\angle BAX = \theta_2$. Therefore, the slope of line DQ (m_1) = tan θ_1 and the slope of line BP (m_2) = tan θ_2.



In triangle EAC, $\angle ECX = \angle AEC + \angle EAC$ ($\because$ the exterior angle of a triangle is equal to the sum of non adjacent interior angles.)

or, $\theta_1 = \theta + \theta_2$

or, $\theta = \theta_1 - \theta_2$

Taking the tangent trigonometric ratio on both sides:

$\tan\theta = \tan(\theta_1 - \theta_2)$

or, $\tan\theta = \dfrac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$

or, $\tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2}$ ....(i) $(\because m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2)$

We know, $\alpha = 180° - \theta$ ($\because$ The sum of adjacent angle is equal to 180°.)

Taking trigonometric ratio tan on the both sides,

$\tan\alpha = \tan(180° - \theta) = -\tan\theta$

$= -\frac{m_1 - m_2}{1 + m_1 m_2}$ ....(ii)

From (i) and (ii)

$\tan\theta = \pm\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$

Therefore, the angle between two straight lines $(\theta) = \tan^{-1} \pm\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$

If the value of tan θ is positive, angle between two lines is an acute angle. Similarly, if the value of tan θ is negative, angle between two lines is an obtuse angle.

Condition for Parallelism of Two Straight Lines

If two straight lines DQ and PB are parallel to each other, then the angle (θ) between them is 0° or 180°. In the adjacent figure, DQ and PB are parallel to each other.

We know, $\tan\theta = \pm\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$

or, $\tan 0° = \pm\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$

or, $0 = \pm\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$

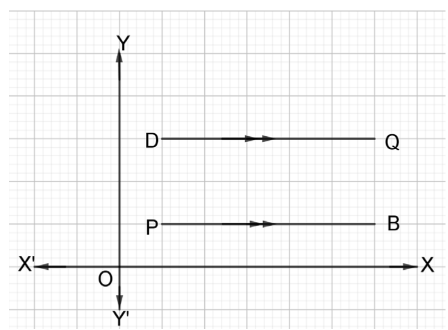
or, $0 \times (1 + m_1 m_2) = \pm(m_1 - m_2)$

or, $0 = \pm(m_1 - m_2)$

or, $0 = m_1 - m_2$

($\because$ Dividing 0 by ±1 gives 0)

or, $m_1 = m_2$



Therefore, if straight lines are parallel to each other, then the slopes of those lines are equal.

Condition for Perpendicular of Two Straight Lines

If two straight lines DQ and PB are perpendicular to each other, then the angle (θ) between them is 90°. In the adjacent figure, DQ and PB are perpendicular to each other.

We know, $\tan\theta = \pm\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$

or, $\tan 90° = \pm\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$

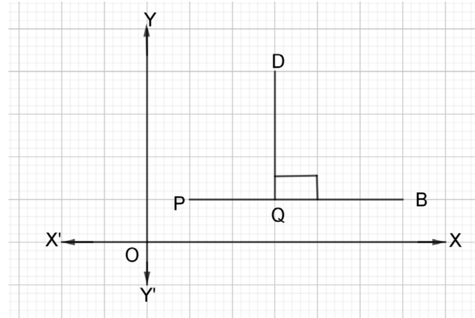
or, $\frac{\sin 90°}{\cos 90°} = \pm\left(\frac{m_1 - m_2}{1 + m_1.m_2}\right)$

or, $\frac{1}{0} = \pm \frac{m_1 - m_2}{1 + m_1 . m_2}$

or, $(1 + m_1m_2) = \pm 0 \times (m_1 - m_2)$

or, $(1 + m_1m_2) = 0$

or, $m_1m_2 = -1$



Therefore, if two lines are perpendicular to each other, the product of their slopes is −1:

or, $m_1 = \frac{-1}{m_2}$

Therefore, the slopes are negative reciprocal to each other.

Activity 1

If two straight lines are in the general form $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, answer the following questions:

a. What are the slopes m_1 and m_2 of the lines?

b. If the angle between these lines is θ, then substituting the values of m_1 and m_2 in $\tan\theta = \pm\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$, what will be the value of θ?

For the first line, slope of $a_1x + b_1y + c_1 = 0$, $(m_1) = -\frac{\text{coefficient of } x}{\text{coefficient of } y} = -\frac{a_1}{b_1}$

For the second line, slope of $a_2x + b_2y + c_2 = 0$, $(m_2) = -\frac{\text{coefficient of } x}{\text{coefficient of } y} = -\frac{a_2}{b_2}$

So, $\tan\theta = \pm\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$

or, $\theta = \tan^{-1} \pm \left(\frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2}\right)$

Activity 2

What is the equation of line parallel to $ax + by + c = 0$? Discuss.

The given equation is $ax + by + c = 0$.........(i)

Find the slope at first: slope $(m_1) = -\frac{\text{coefficient of } x}{\text{coefficient of } y} = \frac{-a}{b}$

Let the equation of line parallel to equation (i) be $y = m_2x + c_1 = 0$..............(ii)

Since the lines represented by the equation (i) and (ii) are parallel, $m_2 = m_1$

or, $m_2 = \frac{-a}{b}$

Substituting the value of m_2 in the equation (ii), $y = \frac{-a}{b}x + c_1 = 0$

or, $by = -ax + bc_1$

or, $ax + by - bc_1 = 0$

or, $ax + by + k = 0$ where $k = -bc_1$

Therefore, equation of line parallel to $ax + by + c = 0$ is $ax + by + k = 0$

Activity 3

Discuss how the equation of the line perpendicular to the line $ax + by + c = 0$ is $bx - ay + k = 0$, and present the conclusion in the classroom.

Relations Related to Two Straight Lines

a. Angle (θ) between two straight lines: $(\theta) = \tan^{-1} \pm \left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$

b. Condition for two straight lines to be parallel: $m_1 = m_2$

c. Condition for two straight lines to be perpendicular: $m_1 m_2 = -1$, $m_1 = \frac{-1}{m_2}$

d. Equation of a line parallel to $ax + by + c = 0$ is $ax + by + k = 0$

e. Equation of a line perpendicular to $ax + by + c = 0$ is $bx - ay + k = 0$

Example 1

The equations of two straight lines are $3x + 4y - 10 = 0$ and $4x - 5y + 2 = 0$.

a. Find the acute angle between the straight lines.

b. Find the obtuse angle between the straight lines.

Solution: Here,

The given equations are: $3x + 4y - 10 = 0$……………………(i)

and $4x - 5y + 2 = 0$………………………(ii)

From equation (i), slope $(m_1) = -\frac{\text{coefficient of } x}{\text{coefficient of } y} = \frac{-3}{4}$

From equation (ii), slope $(m_2) = -\frac{\text{coefficient of } x}{\text{coefficient of } y} = \frac{-4}{-5} = \frac{4}{5}$

a. If angle between the lines is θ then, $\tan\theta = \pm\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$

or, $\tan\theta = \pm\left(\frac{\frac{-3}{4} - \frac{4}{5}}{1 + \left(\frac{-3}{4}\right) \times \frac{4}{5}}\right)$

or, $\tan\theta = \pm\left(\frac{\frac{-3 \times 5 - 4 \times 4}{20}}{\left(\frac{20 - 12}{20}\right)}\right)$

or, $\tan\theta = \pm\left(\frac{-15 - 16}{8}\right)$

or, $\tan\theta = \pm\left(\frac{-31}{8}\right)$

or, $\theta = \tan^{-1} \pm\left(\frac{31}{8}\right)$

or, $\theta = 75.53°$

Therefore, the acute angle between the lines $\theta = 75.53°$

b. Again, the obtuse angle between the lines $\theta = 180° - 75.53° = 104.47°$

Example 2

If the straight lines $ax + 10y + 20 = 0$ and $4x - 8y + 7 = 0$ are parallel to each other, find the value of a.

Solution: Here,

The given equations are: $ax + 10y + 20 = 0$……………………..(i)

and $4x - 8y + 7 = 0$ ………………………..(ii)

From equation (i), the slope $(m_1) = -\frac{\text{coefficient of } x}{\text{coefficient of } y} = \frac{-a}{10}$

From equation (ii), the slope $(m_2) = -\frac{\text{coefficient of } x}{\text{coefficient of } y} = \frac{-4}{-8} = \frac{1}{2}$

Since the given lines are parallel, $m_1 = m_2$

or, $\frac{-a}{10} = \frac{1}{2}$

or, $-2a = 10$

or, $a = -5$

Therefore, the value of a is –5.

Example 3

Find the equation of the line which passes through the point (2, 3) and is perpendicular to the line $3x + 2y - 8 = 0$.

Solution

Let, the equation of the line AB is $3x + 2y - 8 = 0$ ....(i)

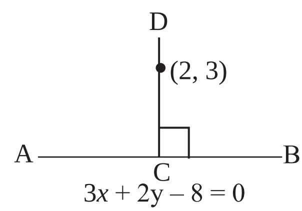
The required line CD passes through the point (2, 3) and is perpendicular to line AB.

From equation (i), the slope of the given line is $(m_1) = -\dfrac{\text{coefficient of } x}{\text{coefficient of } y} = \dfrac{-3}{2}$

Now, equation of line CD passing through (2, 3) is

$y - y_1 = m(x - x_1)$

or, $y - 3 = m(x - 2)$ ....(ii)



Since, slope of line represented by equation (ii) $= m_2$.

The two lines represented by equations (i) and (ii) are perpendicular. So, $m_1.m_2 = -1$

or, $\frac{-3}{2} \times m = -1$

or, $m = \frac{2}{3}$

Now, substituting the value of m in the equation (ii),

$y - 3 = m(x - 2)$

or, $y - 3 = \frac{2}{3}(x - 2)$

or, $3(y - 3) = 2(x - 2)$

or, $3y - 9 = 2x - 4$

or, $2x - 3y + 9 - 4 = 0$

or, $2x - 3y + 5 = 0$

Alternative Method:

Let, the equation of the line is: $3x + 2y - 8 = 0$

The required line CD passes through the point (2, 3) and is perpendicular to line AB.

The line AB is $3x + 2y - 8 = 0$ is $2x - 3y + k = 0$..........(i)

Since this line passes through the point (2, 3), so, substituting in (i)

We get, $2 \times 2 - 3 \times 3 + k = 0$

or, $k = 5$

Substituting $k = 5$ in equation (i), we get,

$2x - 3y + 5 = 0$

Therefore, the required equation is $2x - 3y + 5 = 0$.

Therefore, the required equation is $2x - 3y + 5 = 0$.

Example 4

There are two lines $3x + y = 7$ and $4x - 3y = 5$.

a. Find the intersection point Z of the lines.

b. Find the equation of line passing though the point Z and parallel to the line $2x - y = 3$.

Solution: Here,

a. Let AB and CD be two intersecting lines whose equations are:

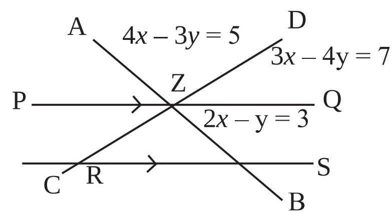
$3x + y = 7$ ....(i) and

$4x - 3y = 5$ ....(ii)

Let, line PQ passes through the point Z and parallel to the line RS is

$2x - y = 3$ ....(iii)

From equation (i) substituting $y = 7 - 3x$ in the equation (ii)



or, $4x - 3(7 - 3x) = 5$

or, $4x - 21 + 9x = 5$

or, $13x = 5 + 21$

or, $x = \frac{26}{13} = 2$

Substituting $x = 2$ in the equation (i), $y = 7 - 3 \times 2 = 7 - 6 = 1$

Therefore, the point of intersection Z is (2, 1).

b. The equation of a line parallel to $2x - y = 3$ is $2x - y + k = 0$ ....(iv)

Since it passes through the point (2, 1) ,

So, $2 \times 2 - 1 + k = 0$

or, $4 - 1 + k = 0$

or, $k = -3$

Substituting $k = -3$ in the equation (iv), $2x - y - 3 = 0$

Therefore, the required equation of the line is $2x - y - 3 = 0$

Example 5

Find equation of perpendicular bisector of the line joining the points (4, 3) and (8, 11).

Solution

Let AB is the line joining (4, 3) and (8, 11) and CD is its perpendicular bisector.

By the formula of midpoint, the coordinates of midpoint of the line AB is

$(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

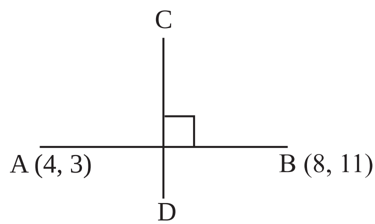
or, $(x, y) = \left(\frac{4 + 8}{2}, \frac{3 + 11}{2}\right)$

or, $(x, y) = \left(\frac{12}{2}, \frac{14}{2}\right) = (6, 7)$

Now, slope of AB $(m_1) = \frac{y_2 - y_1}{x_2 - x_1} = \frac{11-3}{8-4} = \frac{8}{4} = 2$

Let the slope of CD be $(m_2) = m$

By the question, CD is perpendicular to the line AB,



So, $m_1 m_2 = -1$

or, $2m = -1$

or, $m = \frac{-1}{2}$

Now, equation of perpendicular bisector CD passing through midpoint (6, 7) is

$y - y_1 = m(x - x_1)$

or, $y - 7 = \frac{-1}{2}(x - 6)$

or, $2(y - 7) = -1\ (x - 6)$

or, $2y - 14 = -x + 6$

or, $x + 2y - 14 - 6 = 0$

or, $x + 2y - 20 = 0$

Therefore, the equation of perpendicular bisector is $x + 2y - 20 = 0$.

Example 6

Find the equations of the lines that passes through the point (3, 5) and make an angle of 45° with the line $3x - 4y + 15 = 0$.

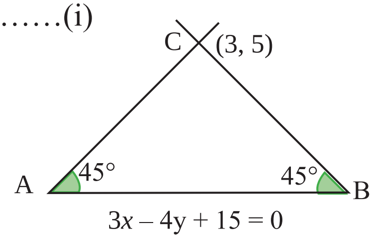
Solution

Let, the line AB having equation: $3x - 4y + 15 = 0$ ....(i)

Lines AC and BC pass through the point (3, 5).

Lines AC and BC make an angle of 45° with line AB.

From equation (i), the slope of line AB



$(m_1) = -\frac{\text{coefficient of } x}{\text{coefficient of } y} = \frac{-3}{-4} = \frac{3}{4}$

Again, the equation of a line passing through (3, 5) is:

$y - 5 = m(x - 3)$ ....(ii)

where the slope of this line $(m_2) = m$.

The angle (θ) between lines (i) and (ii) is given as 45°.

According to the formula: $\tan\theta = \pm\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$

or, $\tan 45° = \pm\left(\frac{\frac{3}{4} - m}{1 + \frac{3}{4}.m}\right)$

or, $1 = \pm\left(\frac{\frac{3-4m}{4}}{\frac{4+3m}{4}}\right)$

or, $1 = \pm\left(\frac{3-4m}{4+3m}\right)$

or, $4 + 3m = \pm(3 - 4m)$

Taking the positive sign (+):

$4 + 3m = (3 - 4m)$

or, $3m + 4m = 3 - 4$

or, $7m = -1$

or, $m = \frac{-1}{7}$

Taking the negative sign (–):

$4 + 3m = -(3 - 4m)$

or, $4 + 3m = -3 + 4m$

or, $3m - 4m = -3 - 4$

or, $-m = -7$

or, $m = 7$

Case 1: $m = \frac{-1}{7}$ Substituting in equation (ii):

$y - 5 = \frac{-1}{7}(x - 3)$

or, $7y - 35 = -x + 3$

or, $x + 7y - 35 - 3 = 0$

or, $x + 7y - 38 = 0$

Case 2: $m = 7$. Substituting in equation (ii):

$y - 5 = 7(x - 3)$

or, $y - 5 = 7x - 21$

or, $7x - y - 21 + 5 = 0$

or, $7x - y - 16 = 0$

Therefore, the required equations are $x + 7y - 38 = 0$ and $7x - y - 16 = 0$.

Exercise 8.2

1. Tick (✓) the correct option for the given questions:

A. Which of the following is the formula to find the angle between two straight lines $y = m_1x + c$ and $y = m_2x + c$

a. $\theta = \tan^{-1}\pm\left(\frac{m_1 + m_2}{1 - m_1 m_2}\right)$

b. $\theta = \tan^{-1}\pm\left(\frac{1 - m_1 m_2}{m_1 - m_2}\right)$

c. $\theta = \tan^{-1}\pm\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$

d. $\theta = \tan^{-1}\pm\left(\frac{1 + m_1 m_2}{m_1 - m_2}\right)$

B. Which of the following is the condition for two straight lines to be parallel?

a. $m_1 - m_2 = 1$ b. $m_1m_2 = -1$ c. $m_1 + m_2 = -1$ d. $m_1 - m_2 = 0$

C. Which of the following is the condition for two straight lines to be perpendicular?

a. $m_1m_2 = 1$ b. $m_1m_2 = -1$ c. $m_1 + m_2 = -1$ d. $m_1 - m_2 = 0$

D. What is the angle between the straight lines $3x - y - 7 = 0$ and $x + 3y - 6 = 0$?

a. 0° b. 30° c. 45° d. 90°

E. What is the acute angle between the straight lines $x - 3y - 2 = 0$ and $2x - y - 1 = 0$?

a. 0° b. 30° c. 45° d. 60°

F. What is the obtuse angle between the straight lines $\sqrt{3}x - y - 5 = 0$ and $y + 10 = 0$?

a. 120° b. 135° c. 150° d. 160°

2. **Find the acute angle between the given lines:**

a. $y - 3x - 4 = 0$ and $y = -2x + 1$ b. $y = 3x + 2$ and $x + 2y - 2 = 0$

3. **Find the obtuse angle between the given lines:**

a. $y = 3x + 4$ and $y = -2x + 1$ b. $2x + 3y - 4 = 0$ and $x + 2y - 3 = 0$

4. **Find the angles between the given lines:**

a. $3x + y - 2 = 0$ and $x + 3y + 12 = 0$ b. $2x - y + 3 = 0$ and $y = \frac{-1}{2}x - 4$

5. **Equations of straight lines are given. Prove that these lines are parallel to each other:**

a. $2x - 3y - 5 = 0$ and $2x - 3y = 12$ b. $x - 5y = 4$ and $3x - 15y + 12 = 0$

6. **Equations of straight lines are given. Prove that these lines are perpendicular to each other:**

a. $y + \sqrt{3}x + 4 = 0$ and $x - \sqrt{3}y = 5$ b. $ax + by + c = 0$ and $bx - ay + c = 0$

7. **If the given lines are parallel to each other, find the values of *a, p, q,* and *m*.**

a. $px + 3y - 12 = 0$ and $4y - 3x + 7 = 0$ b. $2x - 8y + 6 = 0$ and $qx - 12y - 4 = 3$

c. $ax + 3y - 4 = 0$ and $3x + 9y - 5 = 0$ d. $3x + my = 5$ and $\frac{x}{2} + \frac{y}{3} = 1$

8. **If the given lines are perpendicular to each other, find the values of *a, p, q,* and *m*.**

a. $px + 3y - 12 = 0$ and $4y - 3x + 7 = 0$ b. $2x + 3y + 6 = 0$ and $mx - 5y + 20 = 0$

c. $4x + 3y - 7 = 0$ and $3x + qy - 5 = 0$ d. $ax + 5y - 16 = 0$ and $6x + 10y - 9 = 0$

9. Given lines are $px + qy + r = 0$ and $lx - my + n = 0$.

a. Find the slope of the line $px + qy + r = 0$.

b. If the lines $px + qy + r = 0$ and $lx - my + n = 0$ are perpendicular to each other, prove that $pl = qm$.

10. a. Find the equation of the line that passes through the point (3, 4) and is parallel to the line $3x + 4y - 12 = 0$.

b. Find the equation of the line that is parallel to the line joining points (2, 3) and (3, –1) and passes through the point (3, 1).

11. a. Find the equation of the line that passes through the point (2, 5) and is perpendicular to the line $2x + 5y + 31 = 0$.

b. Find the equation of the line that is perpendicular to the line joining points (4, 7) and (2, –3) and passes through the point (3, 1).

12. a. Given lines are $3x + 4y - 7 = 0$ and $5x - 2y - 3 = 0$.

(i) Find the coordinates of the intersection point Z.

(ii) Find the equation of a straight line perpendicular to the line $2x + 3y = 5$.

b. If the straight line $\frac{x}{a} + \frac{y}{b} = 1$ passes through the intersection point of the lines $x + y - 3 = 0$ and $2x - 3y - 1 = 0$ and is parallel to the straight line $x - y - 6 = 0$, find the values of a and b.

13. Find the equation of the perpendicular bisector of the line segment joining the given points:

a. (–2, 3) and (4, 7) b. (4, –2) and (–8, 9) c. (6, –5) and (–8, 9)

14. a. Find the equations of the two lines that pass through the point (3, –2) and make an angle of 45° with the line $2x - 3y + 10 = 0$

b. Find the equations of the two lines that pass through the origin (0, 0) and make an angle of 60° with the line $x + y + 3 = 0$.

Answer

1. A. c　　B. d　　C. b　　D. d　　E. c　　F. a

2. a. 45°　　b. $\theta = \tan^{-1}(7)$　　3. a. 135°　　b. 172.87°

4. a. 53.13°, 126.86°　　b. 90°

5–6. Show to the teacher.　　7. a. $p = -\frac{9}{4}$　　b. $q =$ 3　　c. $a = 1$　　d. $m = 2$

8. a. $p = 4$　　b. $m = \frac{15}{2}$　　c. $q = -4$　　d. $a = -\frac{25}{3}$

9. a. $m_1 = \frac{-p}{q}$　　b. Show to the teacher.　　10. a. $3x + 4y - 25 = 0$　　b. $4x + y - 13 = 0$

11. a. $5x - 2y = 0$　　b. $x + 5y - 8 = 0$　　12. a. $3x - 2y - 1 = 0$　　b. 1, –1

13. a. $3x + 2y - 13 = 0$　　b. $24x - 22y + 125 = 0$　　c. $x - y + 3 = 0$

14. a. $5x - y - 17 = 0$ and $x + 5y + 7 = 0$　　b. $(2 + \sqrt{3})x - y = 0$ and $(2 - \sqrt{3})x - y = 0$

Project Work

Note down two points on each of two roads you see in your community/school or while traveling somewhere (e.g., a main road and a side road). Write the equations of the two lines representing these roads. Then, calculate the angle in degrees between these two roads. Prepare a report including the following points:

(a) Include the actual names of the roads and their photographs.

(b) Describe the process of how the equation of a line was derived from two points.

(c) If the angle between the roads is acute, obtuse, or very large, what impact does it have on road construction planning? If the angle is very small, what kind of problems might arise for traffic safety? Study this and include a brief analysis in the report. Consider the junction of the two roads as the origin. Take 1 meter = 1 unit and note down the coordinates of two points on each road.

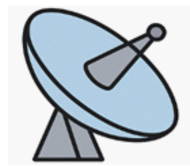
8.3 Conic Section

Activity 1

Study the given figures and answer the following questions:

Figure (a)

Figure (b)



Figure(c)

Figure(d)

- What is the shape of the front surface of the clock in figure (a)?
- What is the shape of the path taken by a planet orbiting the Sun in the solar system in figure (b)?
- What is the geometrical shape of the dish of Nepal Telecom shown in figure (c)?
- What is the geometrical shape the hourglass in figure (d)? Discuss and draw a conclusion.

In the above first figure (a), the visible outer part of the clock is a circle, and the distance from its centre to the circumference is equal in all directions. Does the second figure (b) have properties like a circle? Certainly not. Such a shape is called an ellipse. Similarly, what geometric properties do the Telecom dish in Figure (c) and the sand hourglass in Figure (d) have? The shape like the Telecom dish is called a parabola, and the shape like the sand hourglass is called a hyperbola.

8.3.1 Introduction to Different Parts of Cone

All these shapes can be obtained from a given geometric solid called a right circular cone. Figure (e) shows a right circular cone. The base surface of this cone is circular in shape.

The line perpendicular to the base from the centre of the base to the vertex is called the axis of the cone. In the figure, AB is called the vertical height or axis of the cone. Similarly, AC is called the slant height or generator of the cone. The angle made by the generator (slant height) with the vertical height (axis) is called the semi-vertical angle. In the figure, $\angle BAC = \alpha$ is called the semi-vertical angle.

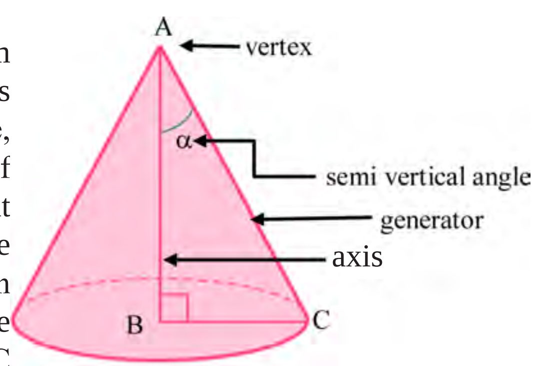


Figure (e)

Introduction to Double-napped Right Circular Cone

If two right circular cones are joined at their vertices, what kind of figure will be formed? Discuss. Also, study the given figure (f)

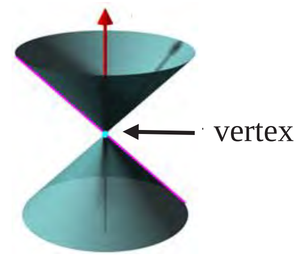


Figure (f)

A solid figure formed by joining the vertices of two equal right circular cones is shown. In this way, two cones joined vertex to vertex are called a double-napped right circular cone.

Circle

Study the given figure (g) alongside.

How has a plane surface ABCD cut a cone? In this way, what geometric shape is formed after cutting? The shape is shown in the figure. Discuss whether the shape formed after cutting is circular or not.

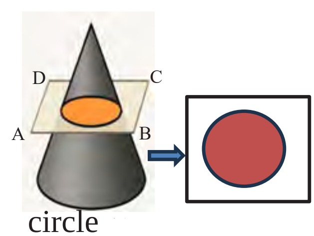


Figure (g)

Therefore, when a plane surface cuts a cone making 90° with the axis or being parallel to the base of the cone, the shape formed (plane section) is a circle.

Ellipse

How has a plane surface ABCD cut a cone? Study the given figure (h). In this way, what geometric shape is formed after cutting? The shape is shown in the figure. Discuss whether the shape formed after cutting is elliptical or not.

When a plane surface cuts one part of a cone such that the angle made by the plane with the axis of the cone is θ , and if the value of θ is greater than the semi-vertical angle (α) and less than 90° ($\alpha < \theta < 90°$), then, in that condition, the plane section formed by the intersection of the cone and the plane is an ellipse.

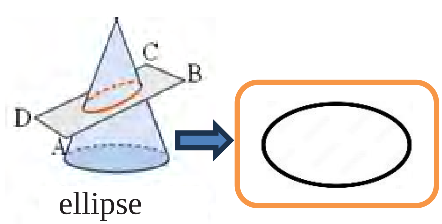


Figure (h)

In the given figure (h), a plane surface ABCD cuts a cone forming an ellipse.

Parabola

How has a plane surface ABCD cut a cone? Study the given figure (i). In this way, what geometric shape is formed after cutting? The shape is shown in the figure. Discuss whether the shape formed after cutting is parabolic or not.

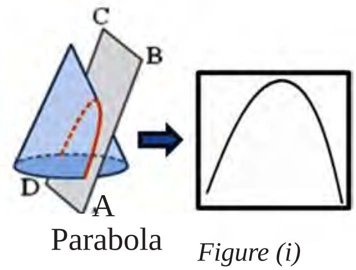


Figure (i)

When a plane surface cuts a cone (intersects it) such that the angle made by the plane with the axis of the cone is θ and is equal to the semi-vertical angle (α = θ), or when the plane surface is parallel to the generator, then, the plane section formed by the intersection of the cone and the plane is called a parabola.

In the given figure (i), a plane surface ABCD cuts a cone forming a parabola. The shape of the parabola formed after cutting is shown in the figure.

Hyperbola

How has a plane surface ABCD cut a doubled mapped right circular cone formed by joining two cones? Study the given figure (j). In this way, what geometric shape is formed after cutting? The shape is shown in the figure. Discuss whether the shape formed after cutting is hyperbolic or not.

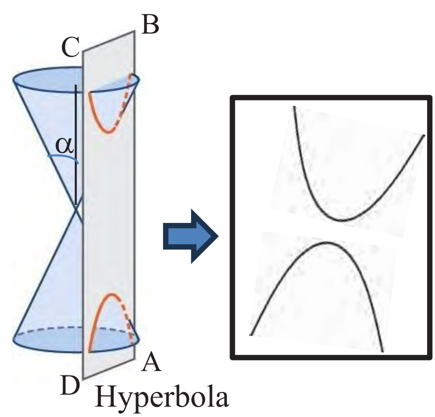


Figure (j)

In the figure, the plane surface ABCD cuts both parts of the cone. In this condition, the angle (θ) made by the plane surface with the axis of the cone is smaller than the semi-vertical angle (*a*), i.e., (θ < α). The plane section formed by the intersection of both parts of the cone and the plane surface is called a hyperbola.

In the given figure (*j*) plane surface ABCD cuts both parts of the cone forming a hyperbola. The shape of the hyperbola formed is shown in the figure.

Thus, to form a hyperbolic plane section, two cones must be joined.

Exercise 8.3

1. Tick (✓) the correct option for the given questions:

A. When a plane surface cuts a cone parallel to its base, which conic section is formed?

a. Parabola b. Ellipse c. Circle d. Hyperbola

B. When a plane surface cuts a cone perpendicular to its axis, not passing through the vertex, which conic section is formed?

a. Parabola b. Ellipse c. Circle d. Hyperbola

C. When a plane surface cuts a right circular cone such that the angle it makes with the axis is less than the semi-vertical angle, which conic section is formed?

a. Parabola b. Ellipse c. Circle d. Hyperbola

D. When a plane surface cuts a cone parallel to its generator, which conic section is formed?

a. Parabola b. Ellipse c. Circle d. Hyperbola

E. If the semi-vertical angle of a cone is α and the angle made by the cutting plane with the axis is θ, and if $\alpha < \theta < 90°$, what is the conic section formed by the intersection?

a. Parabola b. Ellipse c. Circle d. Hyperbola

F. If the semi-vertical angle of a cone is α and the angle made by the cutting plane with the axis is θ, and if $\theta = \alpha$, what is the conic section formed by the intersection?

a. Parabola b. Ellipse c. Circle d. Hyperbola

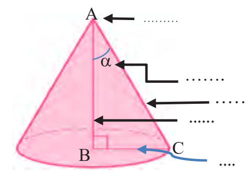
G. If the semi-vertical angle of a cone is α and the angle made by the cutting plane with the axis is θ, and if $\theta < \alpha$, what is the conic section formed by the intersection?

a. Parabola b. Ellipse c. Circle d. Hyperbola

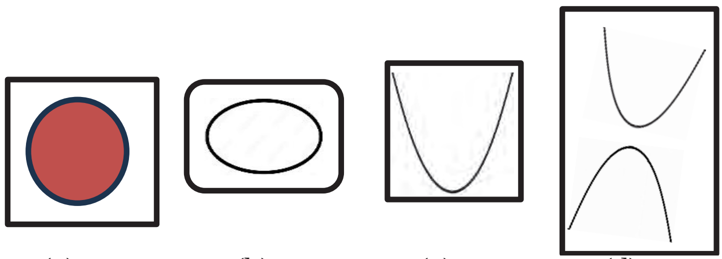
H. Which of the following is not a conic section?

a. Parabola b. Ellipse c. Circle d. Straight line

2. **Write the names of the different parts shown in the given figures.**



3. **Write the names of the following geometrical figures:**



(a) (b) (c) (d)

4. **Define the following conic sections:**

   a. Parabola b. Ellipse c. Circle d. Hyperbola

5. **Answer the following questions:**

   a. Under what condition is a circle formed?

   b. Under what condition is an ellipse formed?

   c. Under what condition is a parabola formed?

   d. Under what condition is a hyperbola formed?

Answer

1. A. d B. c C. c D. b
   E. a F. b G. a H. d

2. Show to the teacher.

3. a. Circle b. Ellipse c. Parabola d. Hyperbola

4. Show to the teacher.

5. a. $\theta = 90°$ b. $\alpha < \theta < 90°$ c. $\alpha = \theta$ d. $\theta < \alpha$

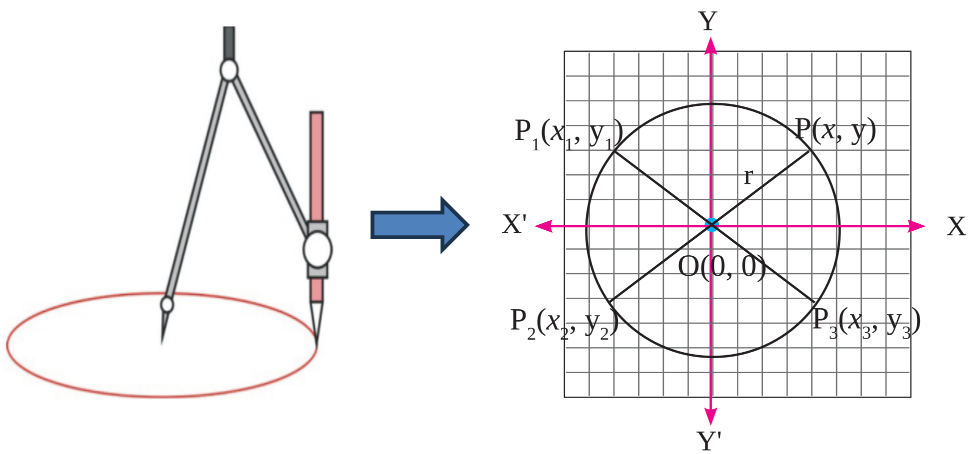
Project Work

Take a soft object like clay, or potato, or carrot, or radish, or dough, and shape it into a cone. How did you cut this object? Different conic shapes (circle, ellipse, parabola, hyperbola) will be formed. Identify the surfaces formed by cutting in this way and draw a diagram of them. Draw a picture and write about it in your notebook. Present this with your report in the class.

8.4 Circle

Activity 1

Using a compass and ruler on graph paper, draw a circle with centre at the origin O(0, 0) as shown in the figure. Take a point P(x, y) on the circumference. Take other points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$, etc., on the circumference. Now, join the centre O and point P(x, y).



The line segment OP is called the radius of the circle, denoted by r. Now, using the distance formula with O(0, 0) = (x_1, y_1) and P(x, y) = (x_2, y_2), fill in the table below:

1	$OP^2 = r^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$	$r^2 = (x - 0)^2 + (y - 0)^2$	$r^2 = x^2 + y^2$
2	$OP_1^2 = r^2 =$	....	....
3	$OP_2^2 = r^2 =$	....	....
4	$OP_3^2 = r^2 =$	....	....

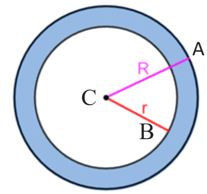
a. Discuss: Are the results obtained from OP^2, OP_1^2, OP_2^2 and OP_3^2 the same?

The relationship $x^2 + y^2 = r^2$ obtained from the table above is called the equation of a circle with centre at the origin (0, 0) and radius r.

b. Measure the lengths OP, OP_1, OP_2 and OP_3 in the figure. Write down their lengths. Are their lengths equal?

....

Activity 2



Draw two circles with the same centre (C) but different radii as shown in the figure. Label them as shown. Let the radius of the inner circle be BC = r and the radius of the outer circle be AC = R.

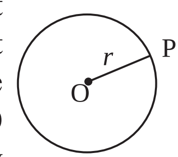
Length of BC (r) = ...	Length of AC (R) = ...

Measure the lengths r and R and compare them. Which one is greater and by how much? Write it. Two circles with the same centre but different radii are called concentric circles. Similarly, draw three concentric circles and compare their radii.

Based on Conic Section:

A shape formed by cutting a right circular cone with a plane surface parallel to its base is also called a circle.

Based on Locus:



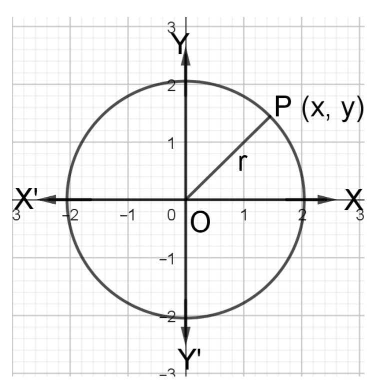
The locus of a point moving at a constant distance from a fixed point is a circle. That fixed point is the centre of the circle, and the constant distance is the radius. In the adjacent circle, O is the centre of the circle. P is a point on the circumference. The distance from point O to point P is called the radius of that circle. The radius is denoted by r. Hence, in the figure, the radius OP = r.

8.4.1 Equation of Circle

a. Equation of a circle with centre at origin (0, 0) and radius r

In the given figure, the centre of the circle is at the origin O(0, 0). Let P(x, y) be a point on the circumference. The distance OP from the centre O to the point P(x, y) on the circumference is the radius of the circle, where OP = r.

Assume O(0, 0) = (x_1, y_1) and P(x, y) = (x_2, y_2)



Now, by the distance formula:

OP = $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

or, $r = \sqrt{(x - 0)^2 + (y - 0)^2}$

or, $r = \sqrt{(x)^2 + (y)^2}$

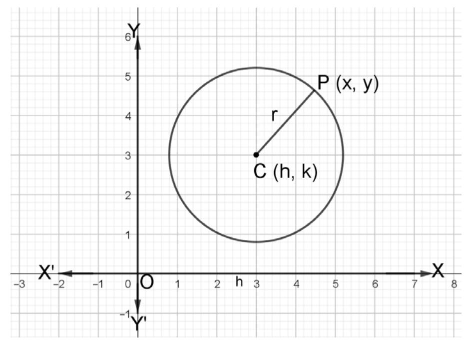
or, $r = \sqrt{x^2 + y^2}$

Squaring on both sides: $\boldsymbol{x^2 + y^2 = r^2}$

Therefore, the equation of a circle with centre at the origin (0, 0) and radius r is $x^2 + y^2 = r^2$.

b. Equation of a circle with centre (*h*, *k*) and radius *r*

In the given figure, point C(*h*, *k*) is the centre of the circle. Let P(*x*, *y*) be a point on the circumference. The distance CP from the centre C to the point P(*x*, *y*) on the circumference is the radius of the circle, where CP = *r*.



Now, assume

$C(h, k) = (x_1, y_1)$

$P(x, y) = (x_2, y_2)$

Now, by the distance formula: $CP^2 = r^2$

or, $(x_2 - x_1)^2 + (y_2 - y_1)^2 = r^2$

or, $(x - h)^2 + (y - k)^2 = r^2$

Therefore, the equation of a circle with centre (*h*, *k*) and radius *r* is

$\mathbf{(x - h)^2 + (y - k)^2 = r^2}$.

c. Equation of a circle having the endpoints of a diameter as (x_1, y_1) and (x_2, y_2)

In the adjacent figure, point C is the centre of the circle. The endpoints of the diameter AB are $A(x_1, y_1)$ and $B(x_2, y_2)$. Let P(*x*, *y*) be a point on the circumference. Join PA and PB.

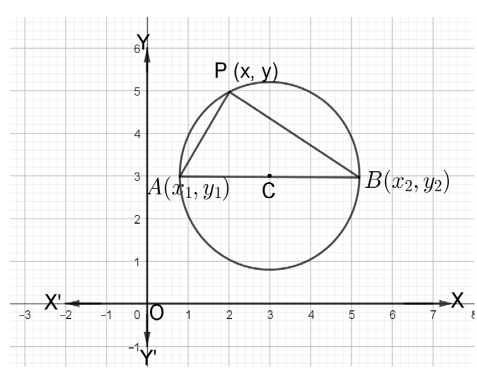
∠APB is an angle inscribed in a semicircle, therefore ∠APB = 90°.

Now, for the slope of line AP, assume $A(x_1, y_1) = (x_1, y_1)$ and $P(x, y) = (x_2, y_2)$.

According to the formula, the slope of AP

$(m_1) = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_1}{x - x_1}$

Similarly, the slope of BP $(m_2) = \frac{y - y_2}{x - x_2}$



We know that AP and BP are perpendicular to each other, so:

$m_1 \times m_2 = -1$

or, $\frac{y - y_1}{x - x_1} \times \frac{y - y_2}{x - x_2} = -1$

or, $\frac{(y - y_1)(y - y_2)}{(x - x_1)(x - x_2)} = -1$

or, $(y - y_1)\ (y - y_2) = -(x - x_1)\ (x - x_2)$

or, $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

Therefore, equation of circle having diameter with end points (x_1, y_1) and (x_2, y_2) is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.

d. General form of equation of circle

The equation of circle with centre (h, k) and radius (r) is $(x-h)^2 + (y-k)^2 = r^2$

or, $x^2 - 2xh + h^2 + y^2 - 2yk + k^2 = r^2$

or, $x^2 + y^2 + 2(-h)x + 2(-k)y + h^2 + k^2 - r^2 = 0$

or, $x^2 + y^2 + 2gx + 2fy + c = 0$ is called general form of equation of circle.

Where, $g = (-h)$, $f = (-k)$ and $c = h^2 + k^2 - r^2$

Where, $h = -g$, $k = -f$ and $r^2 = h^2 + k^2 - c$, $r = \sqrt{h^2 + k^2 - c}$

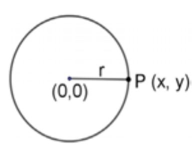
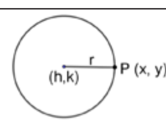
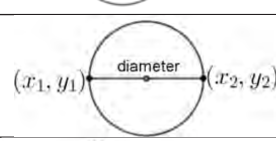
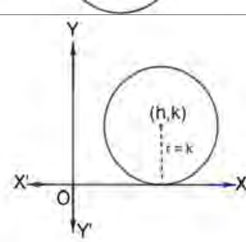
Therefore, centre of circle $(h, k) = (-g, -f)$ and radius $(r) = \sqrt{h^2 + k^2 - c}$

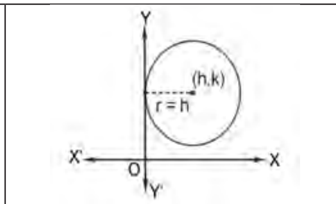
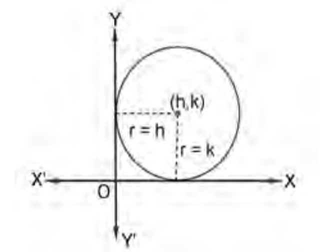
The general equation $x^2 + y^2 + 2gx + 2fy + c = 0$ has following characteristics:

a. The degree of x and y is 2.

b. Coefficient x^2 = Coefficient of y^2

Required Facts

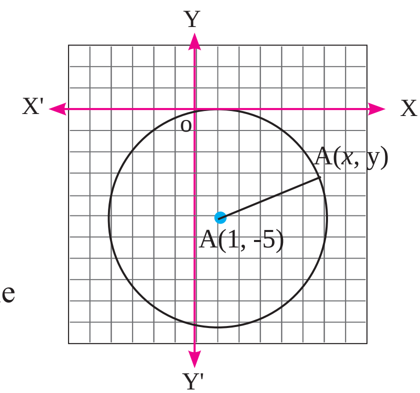
S.N.	Condition of circle	Figures	Equation of circles
1	Centre at origin (0, 0)		$x^2 + y^2 = r^2$
2	Centre at origin (h, k)		$(x-h)^2 + (y-k)^2 = r^2$
3	The end point of diameter are given		$(x - x_1)(x - x_2) + (y - y_1)$ $(y - y_2) = 0$
4	When circle touches X- axis $(r = k)$		$(x-h)^2 + (y-k)^2 = k^2$ Or $(x-h)^2 + (y-r)^2 = r^2$

5	When circle touches Y-axis ($r = h$)		$(x-h)^2 + (y-k)^2 = h^2$ or $(x-r)^2 + (y-k)^2 = r^2$
6	When circle touches both the axes ($r = h = k$)		$(x-h)^2 + (y-h)^2 = h^2$ or $(x-k)^2 + (y-k)^2 = k^2$ or $(x-r)^2 + (y-r)^2 = r^2$

Thought Provoking Question: What happens when a circle touches both the axes and lies on second or third or fourth quadrant? Discuss.

Example 1

Find the equation of the circle with centre (1, –5) and radius 4 units.



Solution: Here,

Centre of the circle $(h, k) = (1, -5)$ and radius $(r) = 4$ units.

According to the formula, the equation of the circle

$(x-h)^2 + (y-k)^2 = r^2$

or, $(x-1)^2 + (y+5)^2 = (4)^2$

or, $x^2 - 2.x.1 + (1)^2 + y^2 - 2.y.5 + (5)^2 = 16$

or, $x^2 - 2x + 1 + y^2 - 10y + 25 = 16$

or, $x^2 + y^2 - 2x - 10y + 1 + 25 - 16 = 0$

or, $x^2 + y^2 - 2x - 10y + 10 = 0$ is the required equation of the circle.

Example 2

If the endpoints of a diameter of a circle are (2, –4) and (–3, 7), find the equation of the circle.

Solution : Here,

Let the endpoints of the diameter be $(2, -4) = (x_1, y_1)$ and $(-3,7) = (x_2, y_2)$.

The equation of the circle having endpoints (x_1, y_1) and (x_2, y_2) of its diameter is:

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

or, $(x - 2)(x + 3) + (y + 4)(y - 7) = 0$

or, $x(x + 3) - 2(x + 3) + y(y - 7) + 4(y - 7) = 0$

or, $x^2 + 3x - 2x - 6 + y^2 - 7y + 4y - 28 = 0$

or, $x^2 + y^2 + x - 3y - 34 = 0$ is required equation of the circle.

Example 3

If the equation of a circle is $9x^2 + 9y^2 - 36x + 6y - 107 = 0$, find the coordinates of its centre and its diameter.

Solution: Here,

Given equation of the circle: $9x^2 + 9y^2 - 36x + 6y - 107 = 0$

Centre coordinates $(h, k) = ?$ and diameter $(d) = ?$

Divide both sides of the equation by 9:

$x^2 + y^2 - 4x + \frac{2}{3}y - \frac{107}{9} = 0$ ....(i)

Thought Provoking Question: Why should both sides be divided by 9?

Comparing equation (i) with the general form $x^2 + y^2 + 2gx + 2fy + c = 0$, we get:

$2g = -4$ or, $g = -2$

$2f = \frac{2}{3}$ or, $f = \frac{1}{3}$ and $c = -\frac{107}{9}$

According to the formula, the centre $(h, k) = (-g, -f) = \left(2, -\frac{1}{3}\right)$

The radius $(r) = \sqrt{h^2 + k^2 - c}$

$= \sqrt{(2)^2 + \left(-\frac{1}{3}\right)^2 - \left(-\frac{107}{9}\right)}$

$= \sqrt{4 + \frac{1}{9} + \frac{107}{9}}$

$= \sqrt{\frac{36 + 1 + 107}{9}} = \sqrt{\frac{144}{9}} = \sqrt{16} = 4$

Diameter $(d) = 2r = 2 \times 4 = 8$ units.

Therefore, the coordinates of the centre are $\left(2, -\frac{1}{3}\right)$ and the diameter is 8 units.

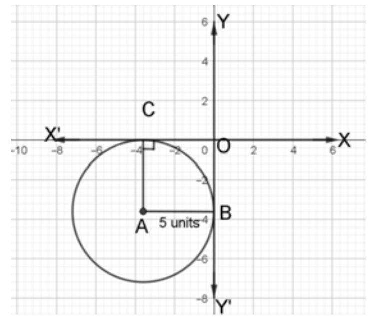
Example 4

Find the equation of the circle that lies entirely in the third quadrant, touches both axes, and has a radius of 5 units.

Solution: Here,

Radius (r) = 5 units

The circle touches both the axes and lies completely on the third quadrant.



So, $(h, k) = (-r, -r) = (-5, -5)$

By the formula, $(x-h)^2 + (y-k)^2 = r^2$

or, $(x+5)^2 + (y+5)^2 = (5)^2$

or, $x^2 + 2.\ x.\ 5 + (5)^2 + y^2 + 2.\ y.\ 5 + (5)^2 = 25$

or, $x^2 + 10x + 25 + y^2 + 10y + 25 = 25$

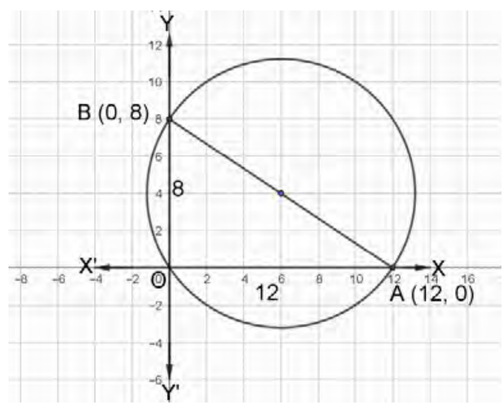
or, $x^2 + y^2 + 10x + 10y + 25 = 0$

Therefore, $x^2 + y^2 + 10x + 10y + 25 = 0$ is the required equation.

Example 5

Find the equation of the circle having the diameter with x- intercept 12 units and y- intercept 8 units.

Solution: Here,



x- intercept = 12 units, y- intercept = 8 units

The circle passes through the origin.

From the figure, $\angle YOX = 90°$

($\because$ the angle between X- axis and Y- axis is 90°)

Therefore, AB is the diameter of the circle,

where A (12, 0) and B (0, 8) are points.

Let, A (12, 0) = (x_1, y_1) and B (0, 8) = (x_2, y_2)

We know, the equation of a circle having the endpoints of a diameter is,

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

or, $(x - 12)(x - 0) + (y - 0)(y - 8) = 0$

or, $(x - 12)x + y(y - 8) = 0$

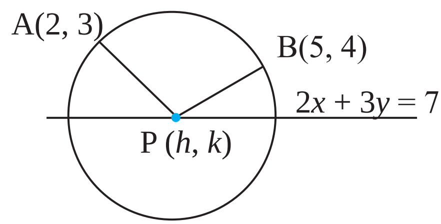
or, $x^2 - 12x + y^2 - 8y = 0$

or, $x^2 + y^2 - 12x - 8y = 0$ is the required equation of the circle.

Example 6

Find the equation of the circle that passes through points A (2, 3) and B (5, 4) and whose centre lies on the straight line $2x + 3y = 7$.

Solution: Here,



Let the centre of the circle be (h, k) and the radius be r.

The centre (h, k) lies on the straight line $2x + 3y = 7$, therefore:

or, $2h + 3k = 7$

or, $h = \frac{7 - 3k}{2}$ ....(i)

Again, points (2, 3) and (5, 4) lie on the circumference of the circle. Therefore, PA = PB (since they are radii of the same circle).

or, $PA^2 = PB^2$

or, $(h - 2)^2 + (k - 3)^2 = (h - 5)^2 + (k - 4)^2$

or, $h^2 - 4h + 4 + k^2 - 6k + 9 = h^2 - 10h + 25 + k^2 - 8k + 16$

or, $13 - 4h - 6k = 41 - 10h - 8k$

or, $10h - 4h = 6k - 8k + 41 - 13$

or, $6h = -2k + 28$

or, $6\left(\frac{7-3k}{2}\right) = -2k + 28$

or, $3(7 - 3k) = -2k + 28$

or, $21 - 9k = -2k + 28$

or, $21 - 28 = 9k - 2k$

or, $7k = -7$

or, $k = -1$

Substituting $k = -1$ in equation (i):

$h = \frac{7 - 3 \times (-1)}{2} = 5$

Now, $PA^2 = r^2 = (5 - 2)^2 + (-1 - 3)^2$

or, $r^2 = 9 + 16 = 25$

$\therefore r = 5$ units

Now, the equation of the circle is: $(x - 5)^2 + (y + 1)^2 = 5^2$

or, $x^2 - 10x + 25 + y^2 + 2y + 1 = 25$

or, $x^2 + y^2 - 10x + 2y + 1 = 0$ is required equation of the circle.

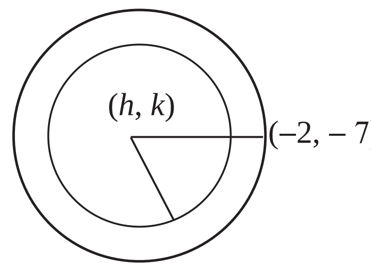
Example 7

Find the equation of the circle concentric with the circle $x^2 + y^2 - 8x + 6y - 5 = 0$ and passing through the point (–2, –7).

Solution: Here,

Given equation: $x^2 + y^2 - 8x + 6y - 5 = 0$……….(i)

and the point is (–2, –7)



Comparing equation (i) with the general form

$x^2 + y^2 + 2gx + 2fy + c = 0$

$2g = -8$, $2f = 6$ and $c = -5$

$g = -4$, $f = 3$ and $c = -5$

According to the formula, the centre $(h, k) = (-g, -f) = (4, -3)$

Let, $(4, -3) = (x_1, y_1)$ and $(-2, -7) = (x_2, y_2)$

By the distance formula, the radius r of the required circle is:

$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$r = \sqrt{(-2 - 4)^2 + (-7 + 3)^2}$

$r = \sqrt{(-6)^2 + (-4)^2}$

$r = \sqrt{36 + 16}$

$r = \sqrt{52}$

$r = 2\sqrt{13}$ units

Now, the equation of the circle is: $(x - h)^2 + (y - k)^2 = r^2$

or, $(x - 4)^2 + (y + 3)^2 = (2\sqrt{13})^2$

or, $x^2 - 8x + 16 + y^2 + 6y + 9 = 52$

or, $x^2 + y^2 - 8x + 6y + 25 - 52 = 0$

or, $x^2 + y^2 - 8x + 6y - 27 = 0$ is the required equation of the circle.

Exercise 8.4

1. Tick (✓) the correct option for the given questions:

A. Which of the following equations represents a circle with centre (0, 0) and radius 10 units?

a. $x^2 - y^2 = 10$ b. $x^2 + y^2 = 10$

c. $x^2 - y^2 = 100$ d. $x^2 + y^2 = 100$

B. Which of the following points lies on the circle with equation $(x - 5)^2 + (y - 4)^2 = 18$?

a. (0, 0) b. (1, 0) c. (2, 1) d. (0, 2)

C. A circle with centre (0, 0) passes through point (–8, –15). What is the radius of the circle?

a. 8 units b. 15 units c. 17 units d. 21 units

D. For the circle with equation $x^2 + y^2 - 4x - 8y - 45 = 0$, which of the following is its centre and radius?

a. Centre (2, 4) and radius 5 units b. Centre (4, 8) and radius 65 units

c. Centre (4, 8) and radius $\sqrt{65}$ units d. Centre (2, 4) and radius $\sqrt{65}$ units

E. The equation of the circle with centre (1, –2) and radius 4 units is:

a. $(x + 1)^2 + (y - 2)^2 = 4^2$ b. $(x + 1)^2 + (y + 2)^2 = 4^2$

c. $(x - 1)^2 + (y - 2)^2 = 4^2$ d. $(x - 1)^2 + (y + 2)^2 = 4^2$

2. Find the equation of the circle with the given centre and radius:

	Centre (h, k)	Radius (r)		Centre (h, k)	Radius (r)
a.	(–4, 1)	4 units	d.	(0, 10)	5 units
b.	(–3, –4)	7 units	e.	(–a, –a)	$a\sqrt{2}$ units
c.	(2, –1)	3 units	f.	(a, b)	b units

3. Find the centre and radius from the given equations of circles:

a. $x^2 + y^2 = 25$ b. $(x - 0)^2 + (y - 0)^2 = 36$

c. $(x - 0)^2 + (y + 2)^2 = 36$ d. $x^2 + y^2 - 2y = 24$

e. $x^2 + y^2 - 10x + 4y + 13 = 0$ f. $x^2 + y^2 + 6x - 8y - 11 = 0$

g. $2x^2 + 2y^2 - 8x - 12y + 1 = 0$ h. $9x^2 + 9y^2 - 36x + 6y - 107 = 0$

4. Find the equation of the circle if the given points are the endpoints of its diameter:

a. $(a, -1)$ and $(-a, 2)$ b. (2, –6) and (3, –7) c. (0, –7) and (–3, 2)

5. **The equation of a circle is $x^2 + y^2 - 4x + 6y - 12 = 0$.**
   a. Find the coordinates of the centre.
   b. If one endpoint of a diameter is (5, 3), find the coordinates of the other endpoint.

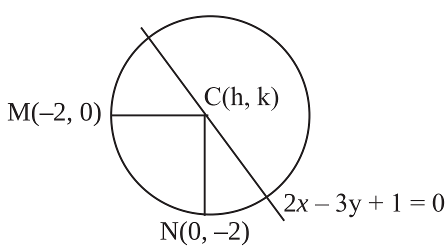
6. **The equation of a circle is $x^2 + y^2 - 6x - 8y - 11 = 0$.**
   a. Find the coordinates of the centre of the circle.
   b. Find the radius of the circle.
   c. If one endpoint of a diameter of this circle is (–2, –3), find the coordinates of the other endpoint.

7. **The equation of a circle is $3x^2 + 3y^2 + 6x - 3y + 3 = 0$.**
   a. Find the coordinates of the centre of the circle.
   b. Find the radius of the circle.
   c. If one endpoint of a diameter of this circle is (5, 3), find the coordinates of the other endpoint.

8. **Find the equation of the circle under the given conditions:**
   a. Lying entirely in the third quadrant, touching both axes, and having radius 5 units
   b. Lying entirely in the fourth quadrant, touching both axes, and having radius 6 units

9. **Find the equation of the circle under the given conditions:**
   a. x- intercept 4 units, y- intercept 6 units, and passing through the origin
   b. x- intercept 5 units, y- intercept 10 units, and passing through the origin
   c. x- intercept 12 units, y-intercept 8 units, and passing through the origin

10. **Find the equation of the circle under the given conditions:**

<table>
<tr>
<td>(a)


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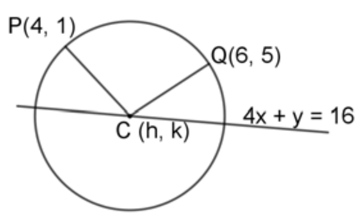
<td>(b)


</td>
</tr>
<tr>
<td>In the figure, the circle passes through points M(–2, 0) and N(0, –2), and its centre C(h, k) lies on the line $2x - 3y + 1 = 0$.</td>
<td>In the figure, the circle passes through points P(4, 1) and Q(6, 5), and its centre C(h, k) lies on the line $4x + y = 16$.</td>
</tr>
</table>

11. Find the equation of the circle passing through the given points:

a. (0, 0), (4, 0) and (0, 2) b. (0, 0), (5, 0) and (0, 5)

12. Find the equation of the circle concentric with the circle $x^2 + y^2 - 8x + 4 = 0$ and passing through the point (4, 2).

13. Find the equation of the circle with centre (–1, 2) that passes through the centre of the circle given by $x^2 + y^2 - 6x - 10y - 2 = 0$.

14. A circular flower garden is built in the centre of the school playground. The centre of the flower bed is taken as (2, –1) in the coordinate system, and its radius is 7 meters.

a. Write the equation of the circle representing the edge of the flower bed.

b. If a student is standing at point (8, –1), determine whether the student is inside, outside, or exactly on the edge of the flower bed. Give a reason.

Answer

1. A. d B. c C. c D. d E. d

2. a. $x^2 + y^2 + 8x - 2y + 1 = 0$ b. $x^2 + y^2 - 2x - 6y - 90 = 0$ c. $x^2 + y^2 - 4x + 2y - 4 = 0$
d. $x^2 + y^2 - 20y + 75 = 0$ e. $x^2 + y^2 + 2ax + 2ay + (2a^2 - 9) = 0$ f. $x^2 + y^2 - 2ax - 2by + a^2 = 0$

3. a. (0, 0), 5 b. (0, 0), 6 c. (0, –2), 6 d. (0, 1), 5 e. (5,–2), 4 f. (–3, 4), 6
g. (2, 3), $\frac{5}{\sqrt{2}}$ h. (2, $\frac{-1}{3}$), 4

4. a. $x^2 + y^2 - y - a^2 - 9) = 0$ b. $x^2 + y^2 - 5x + 13y + 36 = 0$
c. $x^2 + y^2 + 3x + 5y - 14 = 0$

5. a. (2, –3) b. (–1, –9) 6. a. (3, 4) b. 6 unit c. (8, 11)

7. a. (–1, $\frac{1}{2}$) b. $\frac{1}{2}$ units c. (–7, 2)

8. a. $x^2 + y^2 - 10x + 10y + 25 = 0$ b. $x^2 + y^2 - 12x + 12y + 36 = 0$

9. a. $x^2 + y^2 - 4x - 6y = 0$ b. $x^2 + y^2 - 5x - 10y = 0$ c. $x^2 + y^2 - 12x - 8y = 0$

10. a. $x^2 + y^2 - 2x - 2y - 8 = 0$ b. $x^2 + y^2 - 6x - 8y + 15 = 0$

11 a. $x^2 + y^2 - 4x - 2y = 0$ b. $x^2 + y^2 - 5x - 5y = 0$

12. $x^2 + y^2 - 8x + 12 = 0$ 13. $x^2 + y^2 + 2x - 4y - 20 = 0$

14. a. $x^2 + y^2 - 4x + 2y - 44 = 0$ b. Inside

Project work

Find out where equations of circles are used in daily life. Prepare a report by researching with elders, teachers, or using the internet, and present it in the classroom.

Lesson 9 Transformation

9.1 Introduction

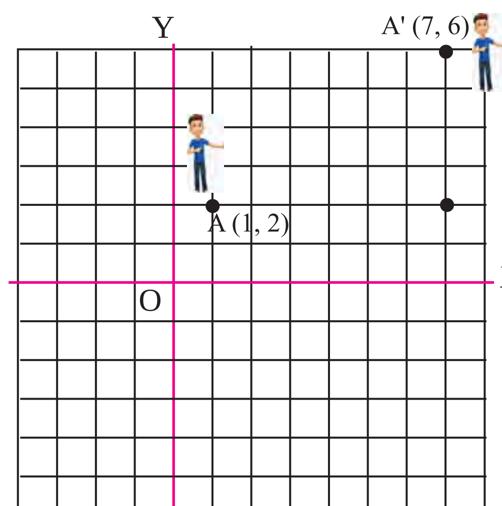
Felix Klein (1849-1925)

The process of changing the position of an object in a given space according to a specific rule is called a transformation. Around 300 BC, the Greek mathematician Euclid, who was born in Alexandria, Egypt, systematically organized geometry in his book Elements, using self-evident facts and proofs. Although Euclid did not directly develop the concept of transformations, his work in geometry provided the German mathematician Felix Klein with the necessary foundation and inspiration to explore transformations. Towards the end of the 19^{th} century, Klein systematically applied transformations as a fundamental aspect of geometry through his Erlangen Program. Transformations are particularly used in mathematics, especially in geometry, calculus and linear algebra. Similarly, the application of transformations can be found in computer science, physics, engineering and music etc.

9.2 Translation of Geometrical Shapes

Activity 1

In the given graph, a person is standing at point A(1, 2). They move straight to the right to reach point B, and from there, move straight up to reach point A'(7, 6). In this situation, discuss the following questions:



a. How many units to the right and how many units up does the person move from point A(1, 2) to reach A'(7, 6)?

b. How can the pair of numbers representing the rightward and upward translation be expressed?

c. How can the relationship between point A(1, 2), the translation pair, and point A'(7, 6) be shown?

From point A(1, 2), the person moves 6 units to the right and 4 units up to reach A'(7, 6). Moving 6 units to the right shifts the position along the X- axis, and moving 4 units up shifts the position along the Y- axis. Therefore, the translation can be represented as the pair $\binom{6}{4}$ or (6, 4). It is known as translation vector. The relationship between point A(1, 2), the translation pair, and point A'(7, 6) can be written as:

$$A(1, 2) \xrightarrow{\binom{6}{4}} A'(1+6, 2+4) = A'(7, 6)$$

In general, if a point P(*x*, *y*) is moved by a translation $\binom{a}{b}$, its new position P'($x + a, y + b$) is obtained. or, $P(x, y) \xrightarrow{\binom{a}{b}} P'(x + a, y + b)$

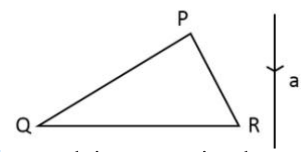
Thought Provoking Question: If the person moves 6 units to the left and 4 units down from point A(1, 2) to reach point A', what will be the coordinates of A'? Discuss and write your answer.

Activity 2

The following questions are to be discussed among friends and present answers in the classroom:

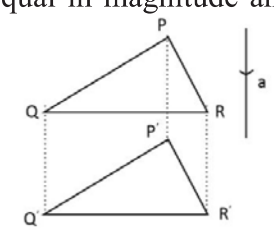
a. Where does ΔPQR reach when it is translated according to the given vector's direction and magnitude?

b. Are ΔPQR and its image after translation congruent?



Here,

a. Using a set square, draw vectors $\overrightarrow{PP'}$, $\overrightarrow{QQ'}$ and $\overrightarrow{RR'}$ equal in magnitude and parallel to $\vec{a}$ from points P, Q, and R, respectively. Then, join the points P', Q', and R' to form ΔP'Q'R', which is the translated image of ΔPQR.



b. ΔPQR and its translated image ΔP'Q'R' are congruent.

Moving any point or object in a given direction by a fixed distance is called translation. For translation, it is necessary to specify both its magnitude and direction. Therefore, translation is a vector. When describing coordinates: A rightward translation is written as (+), A leftward translation as (–), An upward translation as (+), and A downward translation as (–).

Example 1

Find the coordinates of the images when the points A(3, 1) and B(−4, 2) are translated by the translation vector $T = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

Solution: Here,

Here, the given points are A(3, 1) and B(−4, 2). The translation vector is $T = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$
When the translation is applied, the images are obtained.

We know that

$$P(x, y) \xrightarrow{T = \begin{pmatrix} a \\ b \end{pmatrix}} P'(x + a, y + b)$$

Therefore,

$$A(3, 1) \xrightarrow{T = \begin{pmatrix} 1 \\ 4 \end{pmatrix}} A'(3 + 1, 1 + 4) = A'(4, 5)$$

$$B(-4, 2) \xrightarrow{T = \begin{pmatrix} 1 \\ 4 \end{pmatrix}} B'(-4 + 1, 2 + 4) = B'(-3, 6)$$

Hence, the coordinates of the images are A'(4, 5) and B'(−3, 6).

Example 2

Find the coordinates of the points X' and Y', which are the images obtained when the line XY joining X(7, −9) and Y(−1, −1) is translated in the magnitude and direction of the given vector $\overrightarrow{XY}$.

Solution: Here,

Here, X(7, −9) and Y(−1, −1) are two points.

$\overrightarrow{OX} = (7, -9)$ and $\overrightarrow{OY} = (-1, -1)$

Therefore, $\overrightarrow{XY} = \overrightarrow{OY} - \overrightarrow{OX} = (-1, -1) - (7, -9) = (-8, 8)$

Now,

$$X(7, -9) \xrightarrow{\overrightarrow{XY} = \begin{pmatrix} -8 \\ 8 \end{pmatrix}} X'(7 - 8, -9 + 8) = X'(-1, -1)$$

$$Y(-1, -1) \xrightarrow{\overrightarrow{XY} = \begin{pmatrix} -8 \\ 8 \end{pmatrix}} Y'(-1 - 8, -1 + 8) = Y'(-9, 7)$$

Hence, X'(−1, −1) and Y'(−9, 7) are respectively the images of the points X(7, −9) and Y(−1, −1).

Example 3

If M(−4, 3) is translated to M'(4, 4) by a translation vector, find the translation vector. Also, using that translation vector, find the image of N(2, −5).

Solution: Here,

Let the translation vector be $T = \begin{pmatrix} a \\ b \end{pmatrix}$. The points are M(–4, 3) and M'(4, 4).

Now, $M(-4, 3) \xrightarrow{T = \begin{pmatrix} a \\ b \end{pmatrix}} M'(-4 + a, 3 + b)$

But according to the question, M'(4, 4) is image of M(–4, 3).

Therefore, $(4, 4) = (-4 + a, 3 + b)$

Since the corresponding coordinates of the same point are equal, $4 = -4 + a$ and $4 = 3 + b$

$\therefore\ a = 4 + 4 = 8$ and $b = 4 - 3 = 1$

Therefore, the translation vector $T = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \end{pmatrix}$

Again,

$N(2, -5) \xrightarrow{T = \begin{pmatrix} 8 \\ 1 \end{pmatrix}} N'(2 + 8, -5 + 1) = N'(10, -4)$

Hence, the image of N(2, –5) is N'(10, –4).

Example 4

If the vertices of ΔABC are A(−2, 1), B(−2, −4), and C(1, 4), find the coordinates of the image obtained when ΔABC is translated by the vector $T = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. Also represent both triangles in a graph.

Solution: Here,

Here, the vertices of ΔABC are A(−2, 1), B(−2, −4), and C(1, 4).

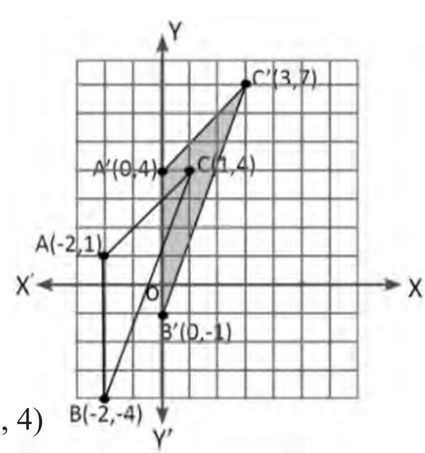
Also, the translation vector $T = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

We know that,

$P(x, y) \xrightarrow{T = \begin{pmatrix} a \\ b \end{pmatrix}} P'(x + a, y + b)$

Therefore,

$A(-2, 1) \xrightarrow{T = \begin{pmatrix} 2 \\ 3 \end{pmatrix}} A'(-2 + 2, 1 + 3) = A'(0, 4)$



$$B(-2, -4) \xrightarrow{T=\binom{2}{3}} B'(-2+2, -4+3) = B'(0, -1)$$

$$C(1, 4) \xrightarrow{T=\binom{2}{3}} C'(1+2, 4+3) = C'(3, 7)$$

Hence, A'(0, 4), B'(0, −1) and C'(3, 7) are the coordinates of the image triangle A'B'C'.

Again, ΔABC and ΔA'B'C' are represented together in the adjacent diagram.

Exercise 9.1

1. Tick (✓) the correct option for the given questions:

A. Which of the following is a translation?

a. Rotating a figure   b. Moving a figure from one place to another

c. Enlarging a figure   d. Making a mirror image of a figure

B. What effect does translation by a vector have on the shape and area of a triangle?

a. Only the shape changes.   b. Only the area changes.

c. Both shape and area change.   d. Both shape and area remain the same.

C. What is the formula to find the image of a point P (x, y) when it is translated by the vector $T = \binom{a}{b}$?

a. $P(x, y) \xrightarrow{T=\binom{a}{b}} P'(x + a, y + b)$

b. $P(x, y) \xrightarrow{T=\binom{a}{b}} P'(x - a, y - b)$

c. $P(x, y) \xrightarrow{T=\binom{a}{b}} P'(x - a, y + b)$

d. $P(x, y) \xrightarrow{T=\binom{a}{b}} P'(x + a, y - b)$

D. If the point Q(−2, 4) is translated by the translation vector $T = \binom{3}{2}$, which of the following is the image Q′?

a. Q'(−1, 6)   b. Q'(1, 6)   c. Q'(−1, −6)   d. Q'(5, 6)

E. If the point P(3, −4) is translated and its image is P'(−2, −1), then, what is the value of the translation vector T?

a. $T = \binom{1}{-3}$ b. $T = \binom{5}{-5}$ c. $T = \binom{-5}{3}$ d. $T = \binom{-1}{-3}$

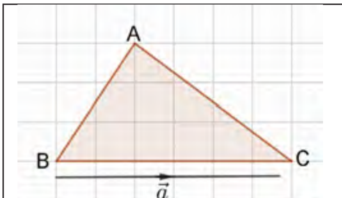
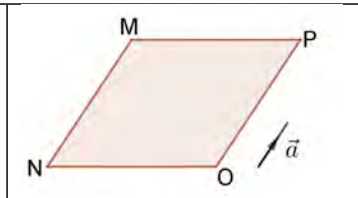
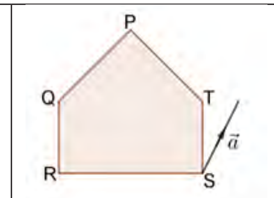
F. If the point P(−1, 4) is translated 5 units to the right along the X- axis, what will be the coordinates of its image?

a. P'(4, 4) b. P'(–6, 4) c. P'(–1, 9) d. P'(4, –4)

G. If a point (x, y) is translated by (–3, 5), which of the following is the resulting point?

a. $(x – 3, y – 5)$ b. $(x + 3, y + 5)$ c. $(x – 3, y + 5)$ d. $(x – 3, y – 5)$

2. Translate the given figures using the given translation vector $\vec{a}$:

		
(a)	(b)	(c)

3. What is meant by translation? Explain with an example.

4. Using the translation vector (T) = $\begin{pmatrix}2\\-2\end{pmatrix}$, find the coordinates of the images of the following points:

a. A(−1, 3) b. B(2, –2) c. C(5, 6)

d. D(−4, −4) e. E(7, 2) f. F(2, –4)

5. If P(2, 3), Q(−2, 4), and R(4, −2) are the vertices of ΔPQR, and it is translated by the translation vector (T) = $\begin{pmatrix}3\\2\end{pmatrix}$ to ΔP′Q′R′, find the coordinates of the vertices of ΔP′Q′R′.

6. Given the points A(3, 5), B(−3, 5), and C(3, −5):

a. To which point is A translated by $\overrightarrow{AB}$?

b. To which point is B translated by $\overrightarrow{BC}$?

c. To which point is C translated by $\overrightarrow{CA}$?

7. If a translation vector transforms A(−5, 6) to A'(−3, 3), find the translation vector.

8. Find the translation vector that moves the point R(4, 7) to R'(–3, –5). Using this translation vector, find the image of the point S(4, 1).

9. If the point U(5, 3) $\xrightarrow{T=\binom{a}{b}}$ U'(4, 2), find the values of a and b for the vector $T = \binom{a}{b}$. Using this vector, find the image of the point V(–5, 4).

10. The vertices of ΔABC are A(1, –1), B(–2, 2), and C(3, 3). If the triangle is translated by the vector $T = \binom{2}{3}$, find the coordinates of the image ΔA'B'C'. Also, represent both ΔABC and Δ A'B'C' on a graph.

11. The points A(1, 8), B(–3, 9), C(0, 13), and D(4, 12) are the vertices of a quadrilateral ABCD.

 a. Find the vector $\overrightarrow{AB}$.

 b. Using vector $\overrightarrow{AB}$. translate the parallelogram ABCD and find the coordinates of its image. Also, represent both parallelograms on a graph.

12. The vertices of triangle ΔABC are A(–4, 6), B(3, –2), and C(1, 2). If the triangle is translated by vector $T_1 = \binom{1}{-2}$ to form Δ A'B'C', and then, translated by vector $T_2 = \binom{-2}{3}$ to form ΔA"B"C" :

 a. Find the coordinates of ΔA'B'C' and ΔA"B"C" .

 b. Represent the triangles ΔABC, Δ A'B'C' and ΔA"B"C" on the same graph.

13. A point P(x, y) is reflected in the line x = 1, and the image is then reflected again in the line x = 3. Compare this final image with the image obtained by translating the point P(x, y) by the vector T = (4, 0). What conclusion can you draw from this? Explain clearly.

Answer

1 – 3. Show to the teacher.

4. a. A'(1, 1) b. B'(4, –4) c. C'(7, 4) d. D'(–2, –6) e. E'(9, 0) f. F'(4, –6)

5. P'(5, 5), Q' (1, 6), R'(7, 0) 6. a. A'(–3, 5) b. B'(3, –5) c. C'(3, 5)

7. $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ 8. $\begin{pmatrix} -7 \\ -12 \end{pmatrix}$, (–3, –11) 9. $a = -1$, $b = -1$, (–6, 3)

10. A'(3, 2), B'(0, 5), C'(5, 6) and show the graph to the teacher.

11. a. $\overrightarrow{AB} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$ b. A'(–3, 9), B'(–7,10), C'(–4, 14), D' (0, 13) and show the graph to the teacher.

12. a. A'(–3, 4), B'(4, –4), C'(2, 0), A'' (–5, 7), B''(2, –1), C''(0, 3)

  b. Show the graph to the teacher.

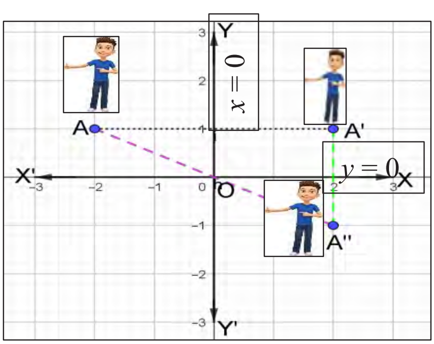
13. Show to the teacher.

9.3 Combined Transformation

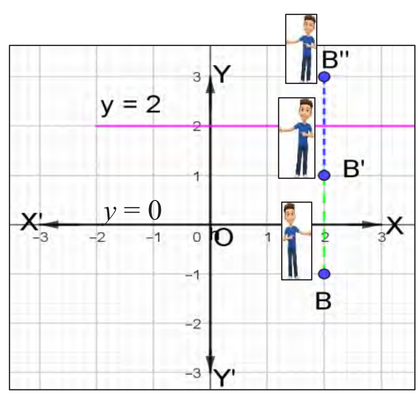
Activity 1

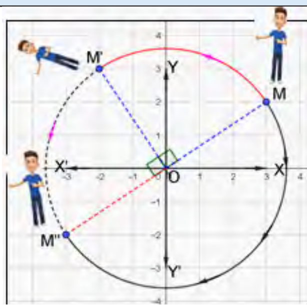
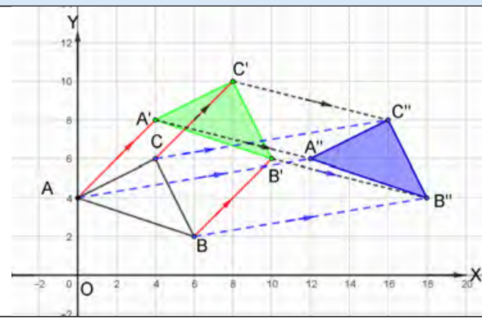
Study the given figures and answer the following questions.

a



b



a. What are the coordinates of A, A' and A"? b. In which line is the point A reflected to obtain its image A'? c. In which line is the point A' reflected to obtain A"? d. Are the two lines of reflection intersecting each other or not? e. Which single transformation can be used to obtain the image A" from the point A? Discuss.	a. What are the coordinates of B, B' and B"? b. In which line is the point B reflected to obtain its image B'? c. In which line is the point B' reflected to obtain B"? d. Are the two lines of reflection parallel or intersecting? e. Which single transformation can be used to obtain the image B" from the point B? Discuss.
c	**d**
	
a. What are the coordinates of M, M' and M"? b. About which point, through how many degrees, and in which direction is the point M rotated to obtain its image M'? c. About which point, through how many degrees and in which direction is the point M' rotated to obtain M"? d. Which single transformation can be used to obtain the image M" from the point M? Discuss.	a. In the given figure, what are the coordinates of triangles ABC, A'B'C' and A"B"C"? b. By how many units to the right and how many units upward should ΔABC be translated to obtain Δ A'B'C'? c. By how many units to the right and how many units downward should ΔA'B'C' be translated to obtain ΔA"B"C"? d. Which single translation transforms ΔABC directly into Δ A"B"C"? Discuss.

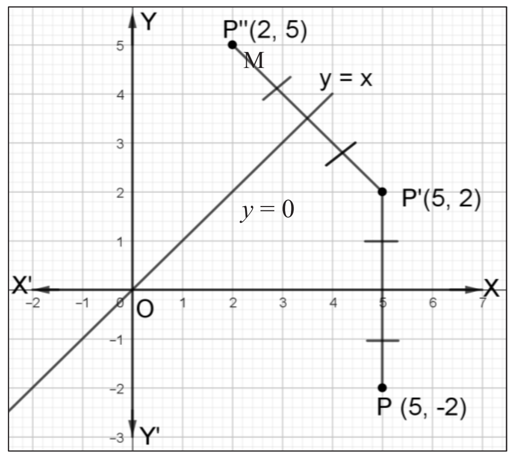
If an object or a geometrical figure is translated successively by two translations r_1 and r_2, moving it from position A to A' and then, from A' to A", then, the single translation that moves it directly from A to A" is called the combined translation.

9.3.1 Combined Transformation of Two Identical Transformation

When two transformations are applied one after another, the new transformation obtained is called a combined (composite) transformation. The combination of two reflections, two rotations, two translations, and two enlargements (dilations) are understood as composite transformations of the respective types.

Combined Reflection

A. When the two reflecting lines intersect at a point.

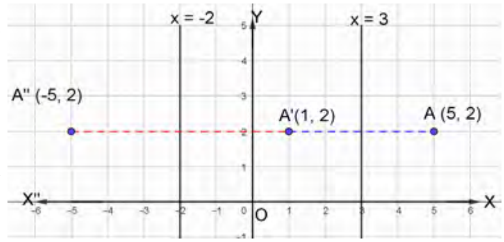


In the given figure, the point P(5, −2) is reflected in the X- axis (the line $y = 0$), and its image becomes P'(5, 2). Again, when the point P'(5, 2) is reflected in the line $y = x$, its image becomes P''(2, 5). Here, the lines $y = 0$ and $y = x$ intersect at the point O. The final image of P(5, −2), which is P''(2, 5), is the same as the image obtained by rotating the point about the origin through +90°. The line $y = x$ makes an angle of 45° with the X- axis. Thus, the composite transformation obtained can be expressed as a rotation about the point of intersection O of the lines $y = x$ and the X- axis ($y = 0$), through an angle equal to twice the angle between the lines. In this figure, the composite transformation that maps P(5, −2) to P''(2, 5) is equivalent to a rotation of 90° about the origin.

In general, When the two reflecting lines intersect at a point, the combined transformation of two successive reflections is equivalent to a rotation through twice the angle between the reflecting lines. The point of intersection of the two reflecting lines is the center of rotation, and the direction of rotation is from the first reflecting line to the second reflecting line. If R_1 and R_2 represent reflections in two intersecting lines, then, their combined transformation is a rotation. The center of this rotation is the point where the lines intersect, and the angle of rotation is 2 × (angle between the reflecting lines).

B. When the two reflecting lines are parallel:

In the given figure, the point A(5, 2) is reflected in the line $x = 3$, and its image becomes A'(1, 2). Again, the point A'(1, 2) is reflected in the line $x = -2$, and its image becomes A''(−5, 2). Here, the lines $x = 3$ and $x = -2$ are parallel to Y- axes. The final image of point A(5, 2) is A''(−5, 2) can be considered a translation, with the distance equal to twice the perpendicular distance between the parallel lines i.e., translation vector $T\begin{pmatrix}a\\b\end{pmatrix} = 2\begin{pmatrix}-2-3\\0\end{pmatrix} = \begin{pmatrix}-10\\0\end{pmatrix}$



When the two reflecting lines are parallel and two successive reflections are performed, the resulting transformation depends on the relative position of the reflecting lines. If the lines intersect, the combined transformation is a rotation about the point of intersection. If the lines are parallel, the combined transformation is equivalent with a translation along the perpendicular direction to the lines.

If R_1 and R_2 represent reflections in the lines $x = h_1$ and $x = h_2$ respectively, and these lines are parallel to the Y- axis, then, the combined transformation of the two reflections is a translation. In this case, discuss how the translation vector can be determined.

$R_2 \circ R_1$ = translation vector = $T\begin{pmatrix}a\\b\end{pmatrix} = 2\begin{pmatrix}h_2 - h_1\\0\end{pmatrix}$ and

$R_1 \circ R_2$ = translation vector = $T\begin{pmatrix}a\\b\end{pmatrix} = 2\begin{pmatrix}h_1 - h_2\\0\end{pmatrix}$

If r_1 and r_2 represent reflections in the lines $y = k_1$ and $y = k_2$ respectively, and these lines are parallel to the X- axis, then, the combined transformation of the two reflections is a translation.

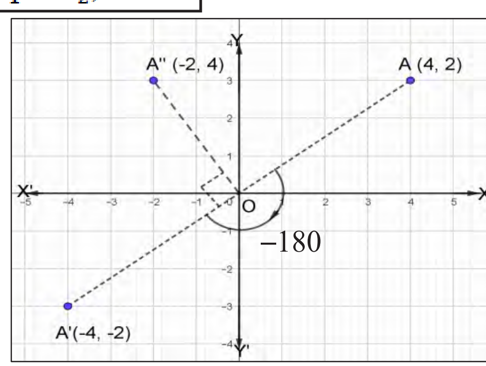
In this case, discuss how the translation vector can be determined.

$r_2 \circ r_1$ = translation vector = $T\begin{pmatrix}a\\b\end{pmatrix} = 2\begin{pmatrix}0\\k_2 - k_1\end{pmatrix}$ and

$r_1 \circ r_2$ = translation vector = $T\begin{pmatrix}a\\b\end{pmatrix} = 2\begin{pmatrix}0\\k_1 - k_2\end{pmatrix}$

Combined Rotation

In the given figure, the point A(4, 2) is rotated about the origin by −180°, giving the image A'(−4, −2). Then, A'(−4, −2) is rotated about the origin again by −90°, giving the final image A''(−2, 4).



If the final image of A(4, 2) is A"(−2, 4), as shown, $P(x, y) \to P'(-y, x)$, it represents a rotation of +90° or −270° about the origin.

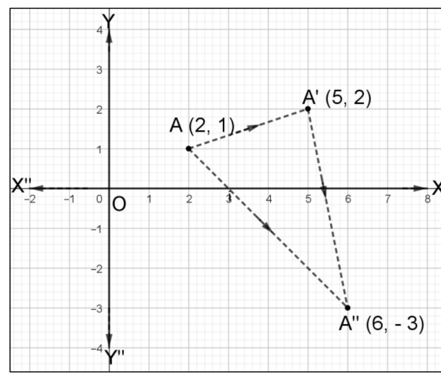
The combined transformation of two rotations having the same center but equal or different angles is equivalent to a single rotation about the same center whose angle is equal to the sum of the two angles. If R_1 and R_2 are rotations about the same center, then, the combined rotation is:

$R_1 \circ R_2 = R_2 \circ R_1 = R_1 + R_2$ i.e., if $R_1[(0, 0), \theta_1]$ and $R_2[(0, 0), \theta_2]$ are two rotations, then their combined rotation is $R_2 \circ R_1 = R_1 \circ R_2 = R[(0, 0), \theta_1 + \theta_2)]$.

Thought Provoking Question: If two rotations $R_1[(0, 0), \theta_1]$ and $R_2[(0, 0), -\theta_2]$ are given, then, what will be the center and the angle of the combined rotation $R_2 \circ R_1$. Discuss.

Combined Translation

In the given figure, when point A(2,1) is translated by the translation vector $(T_1) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ its image becomes A'(5, 2). Similarly, when the image point A'(5, 2) is translated by the translation vector $(T_2) = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$, what will be the coordinates of the image A"? Discuss.



Thus, the final image of point A(2, 1) is A"(6, −3).

We know that,

$A''(6, -3) = (2 + 4, 1 - 4) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 3+1 \\ 1-5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -5 \end{pmatrix}$

Therefore, $\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$

Hence, the combined transformation of two translations is also a translation, where the translation vector is equal to the sum of the two translation vectors. If T_1 and T_2 are two translations, then, their combined translation is: $T_2 \circ T_1 = T_1 \circ T_2 = T_1 + T_2$ i.e., if translation vector $T_1 = \begin{pmatrix} a \\ b \end{pmatrix}$ is followed by $T_2 = \begin{pmatrix} c \\ d \end{pmatrix}$ then the combined translation is denoted by $T_2 \circ T_1$ or simply $T_2 T_1$ where $T_2 \circ T_1 = T_1 + T_2 = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}$ Similarly, $T_{1v} T_2 = T_2 + T_1 = \begin{pmatrix} c \\ d \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c+a \\ d+b \end{pmatrix}$. Thus, the translation vectors $T_1 \circ T_2$ and $T_2 \circ T_1$ are equal.

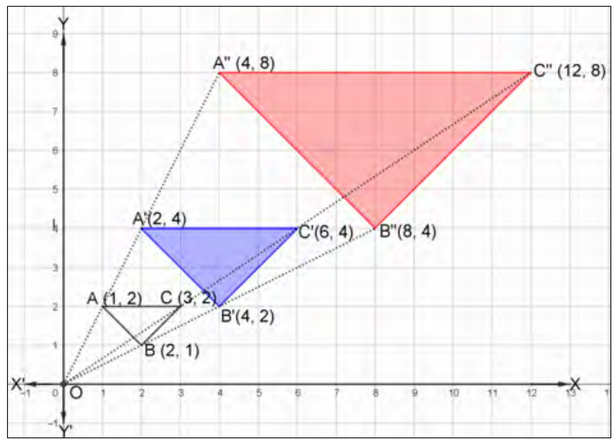
Combined Enlargement

In the given figure, triangle ABC is enlarged to triangle A'B'C' with the centre of enlargement at the origin O(0, 0) and scale factor 2. Similarly, triangle A′B′C′ is again enlarged with the same centre O(0, 0) and scale factor 2. If triangle ABC is enlarged to triangle A'B'C' as (O, 2), then:

a. What is the centre of enlargement when triangle ABC is enlarged to triangle A"B"C"?

b. Similarly, what is the scale factor when triangle ABC is enlarged to triangle A"B"C"? How can it be determined?

c. When triangle ABC is enlarged to triangle A"B"C", discuss what the scale factor is.



From the figure, lines AA", BB", and CC" intersect at the same point O. Therefore, when ΔABC is enlarged to ΔA"B"C", the centre of enlargement remains the origin O(0, 0).

Here, when ΔABC is enlarged to ΔA′B′C′, the scale factor $k_1 = 2$, and when A'B'C' is enlarged to A"B"C", the scale factor $k_2 = 2$. Hence, the combined scale factor is:

$k = k_1 \times k_2 = 2 \times 2 = 4$

Thus, when triangle ABC is enlarged to triangle A"B"C", the enlargement is (O, 4) or E[(0, 0), 4].

In this way, the final image of triangle ABC is triangle A"B"C", which can be represented in coordinate form as follows:

Where, $A(1, 2) \xrightarrow{E[(0,0),4]} A''(4 \times 1, 4 \times 2) = A''(4, 8)$

$B(2, 1) \xrightarrow{E[(0,0),4]} B''(4 \times 2, 4 \times 1) = B''(8, 4)$

$C(3, 2) \xrightarrow{E[(0,0),4]} C''(4 \times 3, 4 \times 2) = C''(12, 8)$

If the centres of enlargement are the same and the scale factors of two enlargements are the same or different, then, the combined transformation of those two enlargements is equivalent to a single enlargement whose centre is the same and whose scale factor is equal to the product of the scale factors of the two enlargements.

a. If two enlargements $E_1[(0, 0), k_1]$ and $E_2[(0, 0), k_2]$ have the same centre at (0, 0), then, the combined enlargement $E_2 \circ E_1 = E_1 \circ E_2 = E[(0, 0), k_1 \times k_2]$

b. If two enlargements $E_1[(a, b), k_1]$ and $E_2[(a, b), k_2]$ have the same centre at (a, b), then, the combined enlargement: $E_2 \circ E_1 = E_1 \circ E_2 = E[(a, b), k_1 \times k_2]$

Example 1

If R_1 represents reflection in the X- axis and R_2 represents reflection in the Y- axis, then, what single transformation is represented by the composite transformation $R_2 \circ R_1$? Also, using that single transformation, find the image of the point A(2, 3).

Solution : Here,

R_1 = reflection in the X- axis

R_2 = reflection in the Y- axis

These reflections intersect at the origin, so the single transformation $R_2 \circ R_1$ is a rotation.

The angle between the X- axis and Y- axis is 90°.

Therefore, the combined rotation

$= R_1 \circ R_2 = 2 \times 90° = 180°$ and the centre of rotation is (0, 0).

Given point A(2, 3).

We know that:

Rotation $R_1 \circ R_2$ about (0, 0) through 180°:

$P(x, y) \xrightarrow{R_1 \circ R_2\ [(0,\ 0), 180°]} P'(-x, -y)$

Now, $A(2, 3) \xrightarrow{R_1 \circ R_2\ [(0,\ 0), 180°]} A'(-2, -3)$

Hence, the image of point A(2, 3) is A'(−2, −3).

Alternatively,

R_1 represents reflection in the X- axis and

R_2 represents reflection in the Y- axis.

Now, $R_1 \circ R_2(2, 3)$
$= R_1(R_2(2, 3))$
$= R_1(-2, 3)$
$= (-2, -3)$

Hence, the image of point A(2, 3) is A′(−2, −3).

Example 2

If R_1 represents reflection in the X- axis and R_2 represents reflection in the line $y = 3$, then, what single transformation is represented by the composite transformation $R_2 \circ R_1$? Also, using that single transformation, find the image of the point B(2, −1).

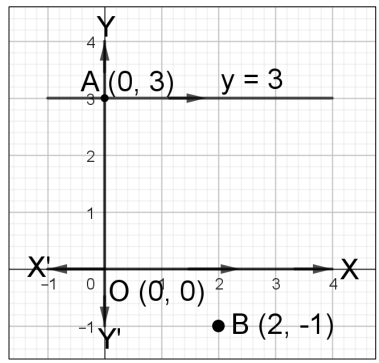
Solution: Here,

R_1 = reflection in the X- axis and R_2 = reflection in the line $y = 3$.

Since R_1 and R_2 are parallel to each other, the composite transformation $R_2 \circ R_1$ is a translation.

Given point B(2, −1),

$$R_2 \circ R_1 = T\begin{pmatrix} a \\ b \end{pmatrix} = 2\,\overrightarrow{OA} = 2\begin{pmatrix} 0-0 \\ 3-0 \end{pmatrix} = 2\begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$$



We know that, $P(x, y) \xrightarrow{T = \begin{pmatrix} a \\ b \end{pmatrix}} P'(x + a, y + b)$

Now,

$B(2, -1) \xrightarrow{T = \begin{pmatrix} 0 \\ 6 \end{pmatrix}} B'(2 + 0, -1 + 6) = B'(2, 5)$

Hence, the image of point B(2, −1) is B′(2, 5).

Example 3

Let, $T_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $T_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ be two translations. Using the composite transformation $T_1 \circ T_2$ or $T_2 \circ T_1$, find the image of the point Q (−3, −4).

Solution: Here,

$T_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $T_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ are two translations. Given point Q (−3, −4). The composite of two translations $T_1 \circ T_2$ or $T_2 \circ T_1$ is also a translation.

The combined translation vector: $T_1 \circ T_2$ or $T_2 \circ T_1 = T_2 + T_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$

Now, the translation vector $T = \begin{pmatrix} a \\ b \end{pmatrix} = T_1 \circ T_2 = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$

According to formula, $P(x, y) \xrightarrow{T = \begin{pmatrix} a \\ b \end{pmatrix}} P(x + a, y + b)$

Now, $Q(-3, -4) \xrightarrow{T = \begin{pmatrix} 5 \\ -3 \end{pmatrix}} Q'(-3 + 5, -4 - 3) = Q'(2, -7)$

Hence, the image of point Q(−3, −4) is Q'(2, −7).

Example 4

$E_1[(0, 0), 2)]$ and $E_2[(0, 0), \frac{3}{2})]$ are two enlargements. Using the composite transformation $E_1 \circ E_2$, find the image of the point M(5,−7).

Solution: Here,

$E_1[(0, 0), 2)]$ and $E_2[(0, 0), \frac{3}{2})]$ are two enlargements. Given point M(5, −7).

We know that the composite transformation of two enlargements is also an enlargement.

The centre of the composite enlargement $E_1 \circ E_2$ is (0, 0)

and the scale factor $(k) = k_1 \times k_2 = 2 \times \frac{3}{2} = 3$

For enlargement with centre (0,0) and scale factor k:

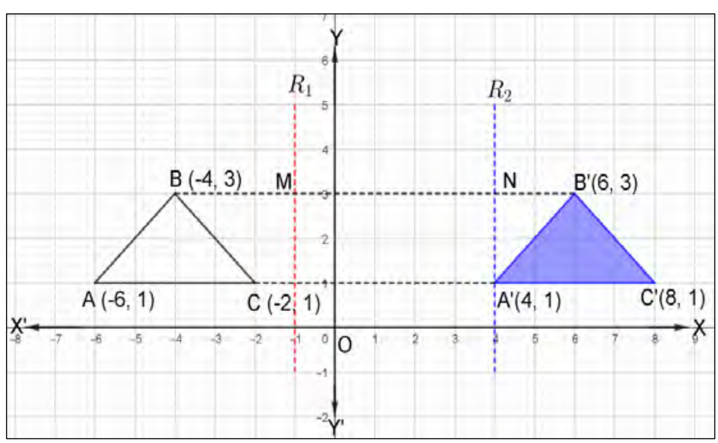
$$P(x, y) \xrightarrow{E[(0,0),\,k]} P'(kx, ky)$$

Now, $M(5, -7) \xrightarrow{E[(0,0),\,3]} M'(3 \times 5, 3 \times (-7)) = M'(15, -21)$

Hence, the image of point M(5, −7) is M′(15, −21).

Example 5

The vertices of a triangle ABC are A(−6, 1), B(−4, 3), and C(−2, 1) . If R_1 represents reflection in the line $x = -1$ and R_2 represents reflection in the line $x = 4$, then, what single transformation is represented by the composite transformation $R_2 \circ R_1$? Using that single transformation, find the image of triangle ΔABC. Also, represent both ΔABC and its image ΔA'B'C' in the same graph.



Solution: Here,

R_1 denotes reflection in the line $x = -1$ and R_2 denotes reflection in the line $x = 4$.

A(−6, 1), B(−4, 3) and C(−2, 1) are the vertices of ΔABC. The lines of reflection $x = -1$ and $x = 4$ are parallel to each other. Therefore, the composite transformation $R_2 \circ R_1$ is equivalent to a translation. The distance between the two lines is MN units.

Thus, $R_2 \circ R_1 = T\begin{pmatrix} a \\ b \end{pmatrix} = 2\,\overrightarrow{MN} = 2\begin{pmatrix} 4+1 \\ 3-3 \end{pmatrix} = 2\begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \end{pmatrix}$

We know that , $P(x, y) \xrightarrow{T = \binom{a}{b}} P'(x + a, y + b)$

Now, $A(-6, 1) \xrightarrow{T = \binom{10}{0}} A'(-6 + 10, 1 + 0)$ = A' (4, 1)

$B(-4, 3) \xrightarrow{T = \binom{10}{0}} B'(-4 + 10, 3 + 0)$ = B' (6, 3)

$C(-2, 1) \xrightarrow{T = \binom{10}{0}} C'(-2 + 10, 1 + 0)$ = C' (8, 1)

Hence, the coordinates of the image of triangle ΔABC i.e., ΔA'B'C', are A'(4, 1), B'(6, 3) and C'(8, 1). ΔABC and its image ΔA'B'C' are shown together in the same graph.

Example 6

The vertices of a triangle PQR are P(4, −2), Q(2, 1), and R(5, 2). If these points are rotated about (0, 0) through 180° and then, again, in the same direction through 90°, what single transformation is represented by the composite transformation? Using that single transformation, find the coordinates of the image of ΔPQR. Also, represent both ΔPQR and its image ΔP'Q'R' in the same graph.

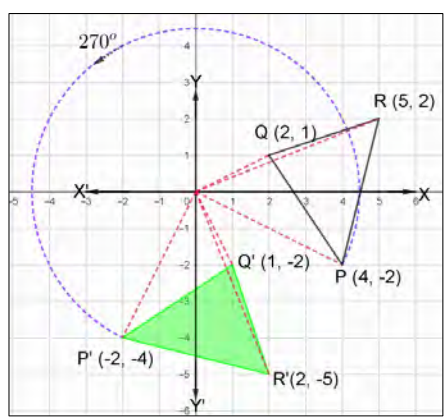
Solution: Here,

The vertices of triangle PQR are P(4, −2), Q(2, 1), and R(5, 2). Let R_1 and R_2 represent rotations about (0, 0) through 180° and 90° respectively in the same direction.

Therefore, the composite transformation $R_2 \circ R_1$ is also a rotation.

$R_2 \circ R_1$ = [(0, 0), 180°+ 90] = [(0, 0), 270°]

We know that:



Applying this transformation,

$P(x, y) \xrightarrow{R[(0,0), 270°]} P(y, -x)$

Now, $P(4, -2) \xrightarrow{R_2 o R_1 [(0,0), 270°]} P'(-2, -4)$

$Q(2, 1) \xrightarrow{R_2 o R_1 [(0,0), 270°]} Q'(1, -2)$

$R(5, 2) \xrightarrow{R_2 o R_1 [(0,0), 270°]} R'(2, -5)$

Hence, the coordinates of the image of ΔPQR, i.e., ΔP'Q'R', are: P'(−2, −4), Q'(1, −2), R'(2, −5). ΔPQR and its image ΔP'Q'R' are presented together in the same graph.

Exercise 9.2.1

1. Tick (✓) the correct option for the given questions:

A. If R_1 represents reflection in the X- axis and R_2 represents reflection in the Y- axis, then, what single transformation is represented by the composite transformation $R_1 \circ R_2$?

a. Rotation [(0, 0), −90°]   b. Rotation [(0, 0), 90°]

c. Rotation [(0, 0), 180°]   d. Rotation [(0, 0), 270°]

B. If R_1 represents reflection in the X- axis and R_2 represents reflection in the Y- axis, then, what is the image of point A(2, 3) under the composite transformation $R_1 \circ R_2$?

a. A'(2, −3)   b. A'(−2, 3)   c. A'(−2, −3)   d. A'(3, 2)

C. If r_1 and r_2 represent rotations about (0, 0) through 45° and 45° respectively in the same direction, then, what is the image of point M(1, 2) under the composite transformation $r_1 o r_2$?

a. A'(−2, 1)   b. A'(1, 2)   c. A'(−1, 2)   d. A'(2, 1)

D. What transformation is obtained by the composition of reflections in the lines $x = 3$ and $x = 5$?

a. Translation by $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ b. Rotation (+180°)

c. Rotation (+90°)   d. Translation by $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$

E. If $R_1[(0, 0), \theta]$ and $R_2[(0, 0), \alpha]$ are rotations, then, what is the composite transformation $R_1 \circ R_2$?

a. $R_1 \circ R_2$ $[(0, 0), \theta - \alpha]$ b. $R_1 \circ R_2$ $[(0, 0), \theta + \alpha]$

c. $R_1 \circ R_2$ $[(0, 0), \theta \times \alpha]$ d. $R_1 \circ R_2$ $[(0, 0), \theta \times \frac{1}{\alpha}]$

F. If translations $T_1 = \begin{pmatrix} m_1 \\ n_1 \end{pmatrix}$ and $T_2 = \begin{pmatrix} m_2 \\ n_2 \end{pmatrix}$ are given, then, what is the composite transformation $T_2 \circ T_1$?

a. $T_2 \circ T_1 = \begin{pmatrix} m_1 - m_2 \\ n_1 - n_2 \end{pmatrix}$ b. $T_2 \circ T_1 = \begin{pmatrix} m_1 + m_2 \\ n_1 + n_2 \end{pmatrix}$

c. $T_2 \circ T_1 = \begin{pmatrix} m_1 \times m_2 \\ n_1 \times n_2 \end{pmatrix}$ d. $T_2 \circ T_1 = \begin{pmatrix} m_1 \times n_2 \\ n_1 \times m_2 \end{pmatrix}$

G. If enlargements $E_1[(a, b), k_1]$ and $E_2[(a, b), k_2]$ are given, then what is the composite transformation $E_1 \circ E_2$?

a. $E_1 \circ E_2[(0, 0), k_1 + k_2]$ b. $E_1 \circ E_2[(0, 0), (k_1 - k_2)]$

c. $E_1 \circ E_2[(0, 0), k_1 \times k_2]$ d. $E_1 \circ E_2[(0, 0), k_1 \times \frac{1}{K_2}]$

2. a. The vertices of a triangle ABC are A(2, 2), B(6, 2), and C(2, 5). First, reflect it in the X- axis and then, in the Y- axis. Find the coordinates of the image obtained by the combined transformation. Also, represent both the original figure and its image in the same graph. What is meant by combined transformation? Write it.

   b. The triangle with vertices A(1, 2), B(4, –1), and C(2, 5) is reflected successively in the lines $x = -3$ and $y = 4$. Present the image ΔA"B"C" obtained from this combined transformation together with triangle ΔABC in the same graph.

   c. The triangle with vertices A(1, 2), B(4, –1), and C(2, 5) is reflected successively in the lines $r_1(x = 4)$ and $r_2(x = -1)$. Find the combined transformation $R_2 \circ R_1$. Present the image ΔA'B'C' obtained by $R_2 \circ R_1$ together with ΔABC in the same graph.

3 a. A triangle ABC has vertices A(4, 0), B(3, 3) and C(1, –1). If these points are rotated about the origin through [(0, 0), 90°] and then, again through [(0, 0),180°] in the same direction, what single transformation does this combined transformation represent? Using that single transformation, find the coordinates of the image A'B'C'. Also, present both triangle ABC and its image in the same graph.

   b. The points P(1, 2), Q(4, –1), and R(2, 5) are the vertices of a triangle PQR. If these points are rotated about the origin through [(0, 0), –90°] and then, through [(0, 0), –180°] in the same direction, what single transformation does this combined transformation represent? Write it. Using that single transformation, find the coordinates of the image. Present both triangles in the same graph.

   c. If R_1 represents a rotation about the origin through 90° and R_2 represents a rotation about the origin through –270°, what does the combined transformation $R_2 \circ R_1$ represent? Using $R_2 \circ R_1$, transform the quadrilateral with vertices A(–4, 0), B(–6, 2), C(–4, 3), and D(2, 5). Present the original and the image in the same graph.

   d. The points A(3, 0), B(4, 2), C(2, 3), and D(1, 2) form a quadrilateral. After rotating it about the origin through 180°, the image is again rotated about the origin through 90° in the positive direction. Present the quadrilateral A'B'C'D' obtained from the combined transformation together with ABCD in the same graph.

4. The points P(2, 1), Q(–1, 4), and R(–2, 2) are the vertices of a triangle ΔPQR. If T_1 and T_2 are two translation vectors: $T_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $T_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

   a. Using the combined transformation $T_1 \circ T_2$ or $T_2 \circ T_1$ find the image of triangle PQR.

   b. Present the ΔPQR and its image ΔP'Q'R' in the same graph.

5. A triangle BUN has vertices B (5, 3), U (1, 4) and N (2, 1). If two translation vectors are $T_1 = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$ and $T_2 = \begin{pmatrix} -1 \\ -4 \end{pmatrix}$,

   a. Find the combined transformation $T_1 \circ T_2$.

   b. Using the combined transformation $T_2 \circ T_1$, find the image of ΔBUN.

   c. Present ΔBUN and its image ΔB′U′N′ in the same graph.

6. The line segment PQ has endpoints P(–3, 5) and Q(7, –4). If $E_1 = [(0, 2)]$ and $E_2 = [0, \frac{3}{2}]$ are two enlargements, and PQ is transformed by $E_1 \circ E_2$, then:

   a. Find the centre of enlargement and the scale factor of the combined transformation $E_1 \circ E_2$.

   b. Find the coordinates of the image line segment P'Q'.

   c. Present PQ and its image P'Q' in the same graph.

7. ΔABC has vertices A(2, 3), B(4, 5), and C(6, 2). If two enlargements are $E_1 = [(0, 0), 4]$ and $E_2 = [(0, 0), \frac{1}{2}]$ then,

   a. Find the scale factor of the combined enlargement $E_2 \circ E_1$.

   b. Using $E_2 \circ E_1$, find the coordinates of the image ΔA'B'C' of ΔABC.

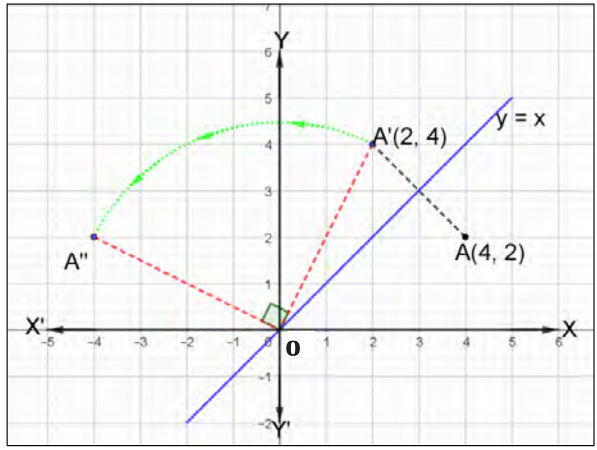
   c. Present ΔABC and its image ΔA'B'C' in the same graph.

Answer

1. A. c B. c C. a D. d E. b F. b G. d
2. a. A''(–2, –2), B''(–6, –2), C''(–2, –5) b. A''(–7, 6), B'' (–10, 9), C''(–8, 3)
   c. A''(–9, 2), B''(–6, –1), C''(–8, 5)
3. a. A''(0, –4), B''(3, –3), C''(–1, –1) b. P''(2, –1), Q''(–1, –4), R''(5, –2)
   c. A''(4, 0), B''(6, –2), C''(4, –3), D''(–2, –5) d. A''(0, –3), B''(2, –4), C''(3, –2), D''(2, –1)
4. a. P'(7, 0), Q'(4, 3), R'(3, 1) b. Show to the teacher.
5. a. $T = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$ b. B'(–3, 2), U'(–4, 1), N'(–2, –1) c. Show the graph to the teacher.
6. a. E[(0, 0), 3] b. P'(–9, 15), Q' (21, –12) c. Show the graph to the teacher.
7. a. $k = 2$ b. A'(4, 6), B'(8, 10), C'(12, 4) and show the graph to the teacher.

9.3.2 Combined transformation of Two Different Transformations

Activity 1



In the given diagram, the point A(4, 2) is reflected in the line $y = x$, and the image obtained is A'(2, 4).

a. In which direction is the image A'(2, 4) rotated about the origin?

b. Through how many degrees must the image A'(2, 4) be rotated to obtain A"?

c. What are the coordinates of the final image A"? Discuss.

d. What single transformation represents the given combined transformation? Discuss and draw a conclusion.

> If F and G represent two different transformations, then:
>
> a. In the combined transformation G∘F, the image obtained after applying transformation F first is then, transformed by G to obtain the final image.
>
> b. In the combined transformation F∘G, the image obtained after applying transformation G first is then, transformed by F to obtain the final image.
>
> c. A single equivalent transformation is identified by studying the relationship between the coordinates of object and the coordinates of final image.

Example 1

A triangle ABC has vertices A(1, 0), B(2, 1) and C(3, –1).

When it is translated by the vector (T) = $\begin{pmatrix}1\\2\end{pmatrix}$, find the coordinates of the image obtained. Then, reflect the obtained image in the line $x = 2$.

Present triangle ABC and the final image A"B"C" in the same graph.

Solution

Here, ABC has vertices A(1, 0), B(2, 1), and C(3, –1).

The translation vector is (T) = $\begin{pmatrix}1\\2\end{pmatrix}$ and the line of reflection is $x = 2$.

According to the formula: $P(x, y) \xrightarrow{T = \begin{pmatrix}a\\b\end{pmatrix}} P'(x + a, y + b)$

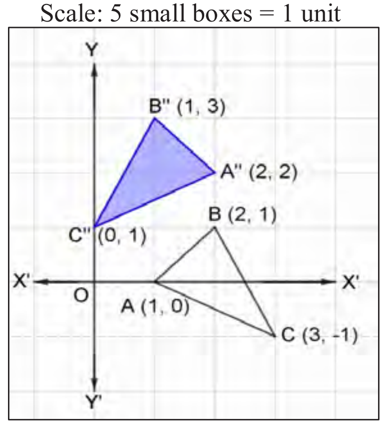
$A(1, 0) \xrightarrow{T = \begin{pmatrix}1\\2\end{pmatrix}} A'(1 + 1, 0 + 2)$ = A' (2, 2)

$B(2, 1) \xrightarrow{T = \begin{pmatrix}1\\2\end{pmatrix}} B'(2 + 1, 1 + 2)$ = B' (3, 3)

$C(3, -1) \xrightarrow{T = \begin{pmatrix}1\\2\end{pmatrix}} C'(3 + 1, -1 + 2)$ = C' (4, 1)

Therefore, after translating ΔABC by the vector (T) = $\begin{pmatrix}1\\2\end{pmatrix}$ So, the image ΔA′B′C′ has coordinates: A'(2, 2), B'(3, 3) and C'(4, 1). Again, reflecting ΔA'B'C' in the line $x = 2$:

Scale: 5 small boxes = 1 unit



$P'(x, y) \xrightarrow{x = h} P''(2 \times h - x,\ y)$

$P'(x, y) \xrightarrow{x = 2} P''(4 - x,\ y)$

$A'(2, 2) \xrightarrow{x = 2} A''(4 - 2,\ 2)$ = A" (2, 2)

$B'(3, 3) \xrightarrow{x = 2} B''(3 - 2,\ 3)$ = B" (1, 3)

$C'(4, 1) \xrightarrow{x = 2} C''(4 - 4,\ 1)$ = C" (0, 1)

Therefore, ΔABC and the final image ΔA"B"C" are presented in the same graph.

Example 2

E represents an enlargement with centre (–3, –4) and scale factor 2, and R represents a reflection in the line $y = 0$. A triangle ABC has vertices A(2, 0), B(3, 1) and C(1, 1).

a. Under the combined transformation E∘R, to which point is P(x, y) mapped?

b. Transform ΔABC using E∘R. Present ΔABC and its image ΔA'B'C' in the same graph.

Solution

Here, the centre of enlargement $(a, b) = (-3, -4)$ and the scale factor $(k) = 2$.

R represents reflection in the line $y = 0$.

a. We know that when the centre of enlargement (a, b) and scale factor (k) are given:

$$P(x, y) \xrightarrow{E[(a,b),k]} P'\{k(x-a)+a, k(y-b)+b\} \text{ and } y = 0$$

And, under reflection in $y = 0$, followed by enlargement:

$$P(x, y) \xrightarrow{y=0} P'(x, -y)$$

Now, $E \circ R(x, y) = E[R(x, y)]$

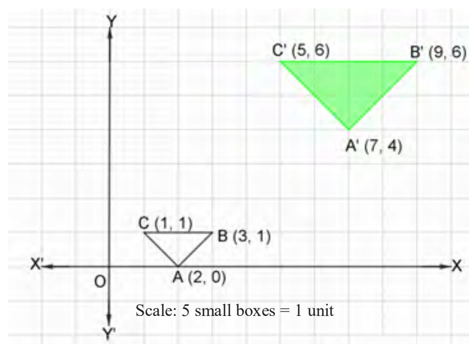
$= E(x, -y)$

$= [2(x-(-3)) - 3, 2(-y-(-4)) - 4)]$

$= 2(x + 3) - 3, 2(-y + 4) - 4)$

$= (2x + 6 - 3, -2y + 8 - 4)$

$= (2x + 3, -2y + 4)$



Hence, $P(x, y) \xrightarrow{E \circ R} P'(2x + 3, -2y + 4)$

b. Therefore: $A(2, 0) \xrightarrow{E \circ R} A'(2 \times 2 + 3, 2 \times 0 + 4) = A'(4 + 3, 0 + 4) = A'(7, 4)$

$B(3, 1) \xrightarrow{E \circ R} B'(2 \times 3 + 3, -2 \times 1 + 4) = B'(6 + 3, -2 + 4) = B'(9, 2)$

$C(1, 1) \xrightarrow{E \circ R} C'(2 \times 1 + 3, -2 \times 1 + 4) = C'(2 + 3, -2 + 4) = C'(5, 2)$

Thus, triangle ΔABC and its final image ΔA'B'C' are presented in the same graph.

Exercise 9.2.2

1. Tick (✓) the correct option for the given questions:

A. Point A(4, 2) is reflected in the line $y = x$ to get A'(2, 4). Then, A' is rotated 90° anticlockwise about the origin. Which single transformation is equivalent to this combined transformation?

a. Reflection in X- axis
b. Reflection in the line $y = x$
c. Reflection in Y- axis
d. Reflection in the line $y = -x$

B. If, under a transformation, the point $P(x, y)$ finally becomes $P''(-y, -x)$, then, which single transformation represents the combined transformation?

$$p\ (x, y) \xrightarrow{R\ (Y-\ axis)} P\ (-x, y) \xrightarrow{R\ [(0,0), +90°]} P''(-y, -x)$$

(a) Reflection in the X- axis (b) Reflection in the Y- axis

(c) Reflection in the line $y = x$ (d) Reflection in the line $y = -x$

C. Reflect the point A(4, –1) first in the line $y = x$. Then, rotate the obtained image A' about the origin through 180°. Which of the following are the coordinates of the final image A"?

a. A"(–1, 4) b. A"(–1, –4) c. A"(1, –4) d. A"(1, 4)

D. The point B (3, 2) is first enlarged by E[(1, 2), –2]. The obtained image is then, translated by the vector (T) = $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.Which of the following are the coordinates of the image B″?

a. B"(–2, 4) b. B"(–2, 3) c. B"(2, –4) d. B"(–3, 2)

2. ΔABC has vertices A(2, –1), B(2, 1), and C(4, –1). First reflect it in the line $y - x = 0$. Then, rotate the obtained image about the origin through 180°. Present the final image ΔA"B"C" and ΔABC in the same graph.

3. a. ΔBIN has vertices B(1, 2), I(–1, –2) and N(5, 0). It is translated by vector (T) = $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and then reflected in the X- axis. Find the coordinates of the image and present ΔBIN and its image in the same graph.

b. ΔMAN has vertices M(–1, 2), A(2, 1) and N(1, 0). It is translated by (T) = $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ Find the coordinates of the image. Then reflect it in the line $x = -2$. Present triangle MAN and its final image ΔM"A"N" in the same graph.

4. a. Let T represent translation and R represent rotation. ΔABC has vertices A(2, 1), B(2, 4), and C(5, 2). It is translated by $\overrightarrow{AB}$ and then rotated 90° anticlockwise about the origin. Find the image according to the combined transformation R∘T. Present ΔABC and its image in the same diagram.

b. ΔABC has vertices A(1, 4), B(4, 1), and C(7, 5). It is translated by $\overrightarrow{AC}$ and then rotated 90° clockwise about the origin. Let T represent translation and R represent rotation. Find the image according to the combined transformation R o T. Present ΔABC and its image in the same graph.

5. a. ΔXYZ has vertices X(2, 3), Y(2, 6) and Z(3, 4). It is translated by a vector $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ and then, enlarged by E[(0, 0), 2]. Find the coordinates of the image and present both the triangle and its image in the same graph.

b. ΔMNP has vertices M(1, 1), N(3, 1) and P(2, 3). Find its image when it is rotated 90° anticlockwise about the origin. Then, enlarge the obtained image with centre (–1, 2) and scale factor 2. Present the final image $\Delta M''N''P''$ and the original triangle MNP in the same graph.

6. a. E represents an enlargement with centre (3, 1) and scale factor 2, and R represents reflection in the line $y = x$. ΔABC has vertices A(2, 3), B(4, 5) and C(1, –2). Find the image of ΔABC under the combined transformation $E \circ R$. Present both diagrams in the same graph.

   b. G_1 represents an enlargement with centre (3, 4) and scale factor 2, and G_2 represents reflection in the line $y = x$. If ΔPQR has vertices P(–3, 5), Q(7, 4) and R(6, –2), then, under the transformation $G_2 \circ G_1$, find the coordinates of P', Q' and R' of $\Delta P'Q'R'$. Present both diagrams in the same graph.

7. a. ΔCUT has vertices C(2, 5), U(–1, 3), and T(4, 1). Rotate ΔCUT 90° clockwise about the origin, then, enlarge it by E [(0, 0), 2]. Find the coordinates of the image and present both the object and image in the same graph.

   b. A parallelogram GITA has vertices G(2, 2), I(6, 2), T(7, 4) and A(3, 4). Rotate it 90° anticlockwise about the origin, then, enlarge it by E[(0, 0), 3]. Find the coordinates of the image and present both the object and image in the same graph.

8. a. Triangle BAT has vertices B(4, 6), A(2, 2), and T(8, 2). It is rotated 90° anticlockwise about the point P(1, 3), then, reflected in the X- axis.

   Find the coordinates of the image of triangle BAT. Present both the object and image in the same graph.

   b. A parallelogram DUCK has vertices D(2, 3), U(1, 1), C(4, 1), and K(5, 3). It is rotated 90° anticlockwise about the point A (4, 0), then, reflected in the line $x = y$. Find the coordinates of the image of parallelogram DUCK. Also, present both the object and image in the same graph.

Answer

1. A. b B. d C. a D. d   2. A''(1, –2), B''(–1, –2), C''(1, –4)

3. a. B'(2, 4), I'(0, 0), N'(6, 2) and B''(2, –4), I''(0, 0), N''(6, –2)

b. M'(0, 4), A'(3, 3), N'(2, 2) and M''(–4, 4), A''(–7, 3), N''(–6, 2)

4. a. A'(–4, 2), B'(–7, 2), C'(–5, 5)   b. A'(5, –7), B'(2, –10), C'(6, –13)

5. a. X'(0, 6), Y'(0, 9), Z'(1, 7) and X''(0, 12), Y''(0, 18), Z''(2, 14)

b. M'(–1, 1), N'(–1, 3), P'(–3, 2) and M''(–1, 0), N''(–1, 4), P''(–5, 2)

6. a. A'(3, 2), B'(5, 4), C'(–2, 1) and A''(3, 3), B''(7, 7), C''(–7, 1)

b. P'(–9, 6), Q'(11, 4), R'(9, –8) and P''(6, –9), Q''(4, 11), R''(–8, 9)

7. a. C'(5, –2), U'(3, 1), T'(1, –4) and C''(10, –4), U''(6, 2), T''(2, –8)

b. G'(–2, 2), I'(–2, 6), T'(–4, 7), A'(–4, 3) and G''(–6, 6), I''(–6, 18), T''(–12, 21), A''(–12, 9)

8. a. B'(–2, 6), A'(2, 4), T'(2, 10) and B''(–2, –6), A''(2, –4), T''(2, –10)

b. D'(1, –2), U'(3, –3), C'(3, 0), K'(1, 1) and D''(–2, 1), U''(–3, 3), C''(0, 3), K''(1, 1)

Project Work

1. In our daily life, many objects that we see or use involve mathematical concepts such as translation and enlargement.
   a. From your home, school, or surrounding environment:
      i. Give at least two examples related to translation.
      ii. Select at least two examples related to enlargement.
   b. For each example:
      i. Draw a diagram showing the initial position of the object.
      ii. Draw the new position after translation or enlargement.
         • If it is translation, mention the direction and distance.
         • If it is enlargement, mention the scale factor.
   c. Compare translation and enlargement.
2. From your daily life surroundings, choose an object or activity where first reflection and then, rotation (or vice versa) can be observed.
   a. Show clearly with diagrams initial position of the object/activity, position after reflection, final position after reflection followed by rotation

b. In the diagram, clearly show the following: Line of reflection (X- axis, Y- axis, or any line), centre of rotation, angle of rotation (such as 90°, 180°)

c. Compare the changes in the position of an object when reflection and rotation are applied separately and when they are applied as a combined transformation. If the order of transformation is changed (i.e., first rotation and then reflection), does the final image remain the same or become different? Explain your answer with reasons based on diagrams.

9.4 Transformation Using Matrix

Activity 1

The process of moving a point or geometrical figure in a fixed direction and distance using matrices is called matrix transformation. Consider a point A(x, y). Translate it using a 2 × 1 matrix $\begin{pmatrix} a \\ b \end{pmatrix}$, and discuss the following questions:

a. What are the coordinates of the new point A' after translation?

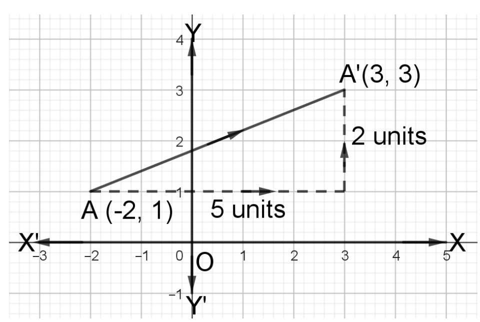
b. Can the point A(x, y) be written in matrix form?

If a point A(x, y) or the vertices of a figure are transformed using a 2 × 1 matrix, it represents a translation. Thus, $A(x, y) \xrightarrow{\binom{a}{b}} A'(x+a, y+b)$. Therefore, the coordinates of A′ become ($x + a$, $y + b$). Also, A (x, y) can be written in matrix form $\begin{pmatrix} x \\ y \end{pmatrix}$, by expressing its coordinates as a column matrix.

9.4.1 Translation Using a 2 × 1 Matrix

In the given diagram, the point A(–2, 1) is translated by a vector $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$. Thus, the image A′ obtained after translation has coordinates:

A'(–2 + 5, 1 + 2) = A'(3, 3)



Object (O) = A(–2, 1) = $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Translation vetor (T) = $\begin{pmatrix} a \\ b \end{pmatrix}$ = $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$

Imave (I) = Object (O) + Translation vector (T)

Image (I) = $\begin{pmatrix} -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

If the coordinates of a point or a geometrical figure are transformed using a matrix, then, that transformation is called a matrix transformation.

If the translation vector (T) = $\begin{pmatrix} a \\ b \end{pmatrix}$ then, when a point P(x, y) is written in column matrix form $\begin{pmatrix} x \\ y \end{pmatrix}$ and translated, the image becomes P′($x + a$, $y + b$). This can be written in matrix form as:

Image (I) = $\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x + a \\ y + b \end{pmatrix}$

Thus, Image (I) = Object (O) + Translation Vector

9.4.2 Transformation Using a 2 × 2 Matrix

Let a 2 × 2 matrix (M) be written in the form (M) = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ When this matrix transforms a point P(x, y), the image becomes P'. A point P(x, y) can be written in column matrix form. If the given column matrix is multiplied by a 2 × 2 matrix, then, we obtain another column matrix $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ Example:

M $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$

or, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$

or, $\begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$

Where, $x_1 = ax + by$ and $y_1 = cx + dy$

Thus, when a 2 × 2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ transforms P(x, y) the matrix form of $\begin{pmatrix} x \\ y \end{pmatrix}$ the image coordinates become ($ax + by$, $cx + dy$).

a. 2 × 2 Transformation Matrix for Reflection in the X- axis

If a point P(x, y) is reflected in the x-axis, then, its image becomes P'(x, $-y$).

How can we find the matrix representing this transformation? Is it always a 2 × 2 matrix? Why is it not a 2 × 1 matrix? Discuss.

Let the image of P(x, y) after reflection in the X- axis ($y = 0$) be P'(x', y').

This can also be written as:

$$P(x, y) \xrightarrow{y = 0} P'(x', y')$$

But in actual reflection in the X- axis:

$$P(x, y) \xrightarrow{y = 0} P'(x, -y)$$

Comparing coordinates of P':

P' (x', y') = P' (x, $-y$)

$x' = x = 1 \times x + 0 \times y$.........(i)

$y' = -y = 0 \times x + (-1) \times y$.........(ii)

Expressing equations (i) and (ii) in matrix form:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Thus, the required 2 × 2 transformation matrix representing reflection in the x-axis (y = 0) is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ obtained. Here, the matrices $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ represent respectively the image matrix, transformation matrix, and object matrix.

Therefore, the image matrix (I) can be written as: I = M × O where M is the transformation matrix and O is the object matrix.

Thought Provoking Questions : What are the required 2 × 2 transformation matrices representing: reflection in the Y- axis ($x = 0$), reflection in the line $y = x$, reflection in the line $y = -x$, reflection in the line $x = h$, and reflection in the line $y = k$?

b. 2 × 2 transformation matrix representing rotation of 90° in the positive direction about the origin:

If a point P(x, y) is rotated 90° anticlockwise about the origin, then, its image becomes P'($-y$, x).

How can we find the matrix representing this transformation? Discuss.

Let the image of point P(x, y) after rotation about the origin by +90° be P'(x', y').

This can be written as:

$$P(x, y) \xrightarrow{[(0,\,0),\,+90^\circ]} P'(x', y')$$

But actually, when a point P(x, y) is rotated by +90° about the origin,

$$P\,(x, y) \xrightarrow{[(0,\,0),\,+90^\circ]} P'(-y, x)$$

Comparing the coordinates of the image point P':

P' (x', y') = P' ($-y$, x)

$x' = -y = 0 \times x + (-1) \times y$.........(i) $y' = x = 1 \times x + 0 \times y$.........(ii)

Expressing equations (i) and (ii) in matrix form:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Thus, the required 2 × 2 transformation matrix representing a rotation of +90° about the origin $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is obtained.

Thought Provoking Question: What are the required 2 × 2 transformation matrices representing: a rotation of –90° (clockwise) about the origin, and a rotation of 180° about the origin? Discuss.

c. 2 × 2 Transformation Matrix for Enlargement with centre (0, 0) and scale factor (*k*)

If a point P(*x*, *y*) is enlarged with centre (0, 0) and scale factor (*k*), then, its image becomes P′(*kx*, *ky*). How can we find the matrix representing this transformation? Discuss.

Let the image of P(*x*, *y*) after enlargement with centre (0, 0) and scale factor *k* be P′(*x*′, *y*′).

This can also be written as:

$$P(x, y) \xrightarrow{E\,[(0,\,0),\,k]} P'(x', y')$$

But in actual form: a point P(*x*, *y*) is enlarged with centre (0, 0) and scale factor (*k*),

$$P(x, y) \xrightarrow{E\,[(0,\,0),\,k]} P'(kx, ky)$$

Comparing coordinates of image of P': P'(*x*', *y*') = P'(*kx*, *ky*)

$x' = kx = k \times x + 0 \times y$.........(i) $\qquad$ $y' = ky = 0 \times x + k \times y$.........(ii)

Writing equations (i) and (ii) in matrix form:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Thus, the required 2 × 2 transformation matrix for enlargement with centre (0, 0) and scale factor k is: $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ where, $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Here, the matrices are called respectively: Image matrix, transformation matrix , object matrix

Details of 2 × 2 matrices in different transformations

For different transformations, the corresponding 2 × 2 transformation matrices are used.

In general: If 2 × 2 transformation matrices are used, then:

Image matrix $(I)_{2 \times n}$ = Transformation matrix $(M)_{2 \times 2}$ × Object matrix

Here, n represents the number of vertices of the figure (or points) involved.

For example: A line segment has *n* = 2 points, A triangle has *n* = 3 points, A quadrilateral has *n* = 4 points.

Thus, whenever you apply a 2 × 2 transformation matrix, you multiply the matrix with each vertex (column) of the object to get the corresponding vertex of the transformed figure. This is the standard procedure in matrix-based transformations for geometric figures.

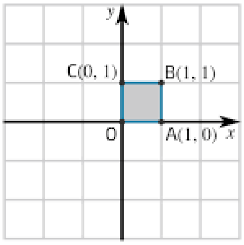
The details of different 2 × 2 transformation matrices used in various transformations are given in the following table.

S.N.	Transformation	Object point	Image point	2 × 2 transformation matrix
1	Reflection on X- axis	P (x, y)	P' $(x, -y)$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
2	Reflection on Y- axis	P (x, y)	P' $(-x, y)$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
3	Reflection on line X = Y or $y - x = 0$ line	P (x, y)	P' (y, x)	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
4	Reflection on $y = -X$ or $x + y = 0$ line	P (x, y)	P' $(-y, -x)$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
5	Rotation about the origin (0, 0) by +90° is the same as rotation about (0, 0) by –270°.	P (x, y)	P' $(-y, x)$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
6	Rotation about the origin (0, 0) by –90° is the same as rotation about (0, 0) by +270°.	P (x, y)	P' $(y, -x)$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
7	Rotation about the origin (0, 0) by ±180°	P (x, y)	P' $(-x, -y)$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
8	Rotation about the origin (0, 0) by ±360°	P (x, y)	P' (x, y)	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
9	Enlargement with center at the origin (0, 0) and scale factor k.	P (x, y)	P' (kx, ky)	$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

Thought Provoking Question: What would the transformation matrix be when a dilation has its center at (a, b) and a scale factor k? Discuss.

9.3.3 Introduction to Unit Square

To introduce a unit square, we consider a square whose side length is 1 unit. For example, consider the square OABC with vertices: O(0, 0), A(1, 0), B(1, 1), C(0, 1) writing its vertices in matrix form, $\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$.



Example 1

Point A(a, b) is mapped to point A($a + 2$, $b - 3$), find the 2 × 1 transformation matrix. Using the same matrix, find the image of point (5, 7).

Solution: Here,

Here, let the 2 × 1 transformation matrix be = $\begin{pmatrix} p \\ q \end{pmatrix}$

$$A(a, b) \xrightarrow{(T) = \begin{pmatrix} a \\ b \end{pmatrix}} A'(a + p, b + q)$$

The given mapping is:

$$A(a, b) \xrightarrow{(T) = \begin{pmatrix} a \\ b \end{pmatrix}} A'(a + 2, b - 3)$$

Equating the corresponding elements of the matrix, we get:

or, $a + 2 = a + p$,

$\therefore\ p = 2$

Similarly, for the second point mapping: $b - 3 = b + q$

$\therefore\ q = -3$

Thus, the 2 × 1 transformation matrix (M) = $\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

We know that the image of any point can be found by multiplying the transformation matrix (TM) with the position vector of the point:

$$\text{Image (I)} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} + \begin{pmatrix} 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

Hence, the image of point (5, 7) can be found as a image $\begin{pmatrix} 7 \\ 4 \end{pmatrix}$ using the same matrix.

Example 2

Find which transformation is represented by the given matrix = $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

Solution: Here,

Let the given transformation matrix be M = $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ and the object be (O) = $\begin{pmatrix} x \\ y \end{pmatrix}$

We know, Image (I) = Transformation matrix (M) × Object (O)

$$= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (0)x + (-1)y \\ (-1)x + (0)y \end{pmatrix} = \begin{pmatrix} -y \\ -x \end{pmatrix}$$

$$\text{Thus, } P(x, y) \xrightarrow{\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}} P'(-y, -x)$$

Therefore, the given matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ represents a reflection about a line $y = -x$.

Example 3

Which transformation does a matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ represent? Using the matrix, find the image of the point A(6, –2).

Solution: Here,

Here, in the first case, let the given transformation matrix be (TM) = $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and the object be (O) = $\begin{pmatrix} x \\ y \end{pmatrix}$.

We know that, Image (I) = Transformation Matrix (TM) × Object (O)

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \times x + (-1)y \\ 1 \times x + (0)y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

Thus, $P(x, y) \xrightarrow{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} P'(-y, x)$

Therefore, the given matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ represents a rotation of +90° (anticlockwise) about the origin.

Now, in the second case, writing the given point A (6, –2) in matrix form: $\begin{pmatrix} 6 \\ -2 \end{pmatrix}$

Image (I) = $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 6 \\ -2 \end{pmatrix}$

Image (I) = $\begin{pmatrix} 0 \times 6 + (-1)(-2) \\ 1 \times 6 + 0 \times (-2) \end{pmatrix}$

Image (I) = $\begin{pmatrix} 0 + 2 \\ 6 + 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$

Hence, the image of the point A(6, –2) is A'(2, 6).

Example 4

Find the 2 × 2 transformation matrix which represents reflection in the line $y = -x$.

Solution: Here,

Here, let a point P(x, y) be reflected in the line $y = -x$ and its image be P'(x', y').

This can also be written as:

$P(x, y) \xrightarrow{y = -x} P'(x', y')$

But actually, when the point P(x, y) is reflected in the line $y = -x$, then:

$P(x, y) \xrightarrow{y = -x} P'(-y, -x)$

Comparing the coordinates of P′,

P'(x', y') = P'(–y, –x)

$x' = -y = 0 \times x + (-1) \times y$.........(i)

$y' = -x = (-1) \times x + 0 \times y$.........(ii)

Writing equations (i) and (ii) in matrix form:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Hence, the required 2 × 2 transformation matrix representing reflection in the line y = –x is: $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

Example 5

If the point P(*a*, *b*) is transformed by matrix $\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$ to P'(–10, –8), find the values of *a* and *b*.

Solution: Here,

$P = \begin{pmatrix} a \\ b \end{pmatrix}$

Transformation Matrix (TM) = $\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$

Image (P') = $\begin{pmatrix} -10 \\ -8 \end{pmatrix}$

We know,

Image(P')=Transformation Matrix (TM) ×Point (P)

or, $\begin{pmatrix} -10 \\ -8 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} \times \begin{pmatrix} a \\ b \end{pmatrix}$

or, $\begin{pmatrix} -10 \\ -8 \end{pmatrix} = \begin{pmatrix} -2b \\ -2a \end{pmatrix}$

Equating corresponding value,

or, $-10 = -2b$, × $b = 5$

or, $-8 = -2a$, × $a = 4$

Therefore, $a = 4$ and $b = 5$

Example 6

The vertices of triangle PQR are P(4, 3), Q(6, 4) and R(8, 1). Using a 2 × 2 transformation matrix $\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$, find the coordinates of the image triangle P'Q'R'.

Solution: Here,

The vertices of triangle PQR are P(4, 3), Q(6, 4) and R(8, 1), and 2 × 2 transformation matrix is $\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$.

Writing these vertices in matrix form (object O) = $\begin{pmatrix} 4 & 6 & 8 \\ 3 & 4 & 1 \end{pmatrix}$

Let the transformation matrix be (TM) = $\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$

We know that:

Image (I) = Transformation Matrix (TM) × Object (O)

$$\text{Image (I)} = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} \times \begin{pmatrix} 4 & 6 & 8 \\ 3 & 4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \times 4 + 3 \times 3 & 1 \times 6 + 3 \times 4 & 1 \times 8 + 3 \times 1 \\ 0 \times 4 + 2 \times 3 & 0 \times 6 + 2 \times 4 & 0 \times 8 + 2 \times 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 + 9 & 6 + 12 & 8 + 3 \\ 0 + 6 & 0 + 8 & 0 + 2 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & 18 & 11 \\ 6 & 8 & 2 \end{pmatrix}$$

Hence, the coordinates of the image ΔP'Q'R' are: P'(13, 6), Q'(18, 8) and R'(11, 2).

Example 7

The vertices of ΔABC are A(1, 2), B(4, 1) and C(2, 5) are transformed into ΔA'B'C' whose vertics are A'(5, 2), B'(6, 1) and C'(12, 5). Find the required 2 × 2 transformation matrix.

Solution: Here,

Let the transformation matrix be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Writing the vertices of triangle ABC, A(1, 2), B(4, 1) and C(2, 5), in matrix form

Object (O) = $\begin{pmatrix} 1 & 4 & 2 \\ 2 & 1 & 5 \end{pmatrix}$

Writing the coordinates of the image ΔA'B'C', where A'(5, 2), B'(6, 1) and C'(12, 5), in matrix form,

Image (I) = $\begin{pmatrix} 5 & 6 & 12 \\ 2 & 1 & 5 \end{pmatrix}$

We know that, Image (I) = Transformation Matrix (TM) × Object (O) or, $\begin{pmatrix} 5 & 6 & 12 \\ 2 & 1 & 5 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} 1 & 4 & 2 \\ 2 & 1 & 5 \end{pmatrix}$

or, $\begin{pmatrix} 5 & 6 & 12 \\ 2 & 1 & 5 \end{pmatrix} = \begin{pmatrix} a+2b & 4a+b & 2a+5b \\ c+2d & 4c+d & 2c+5d \end{pmatrix}$

Equating corresponding elements:

$a + 2b = 5$

or, $a = 5 - 2b$ ....(i)

$4a + b = 6$

or, $4(5 - 2b) + b = 6$ ($\because$ from equation (i))

or, $20 - 8b + b = 6$

or, $-7b = 6 - 20$

Hence, $b = \frac{-40}{-7} = 2$

Substitute the value of b to equation (i):

$a = 5 - 2b = 5 - 2 \times 2 = 1$

Again, $c + 2d = 2$

or, $c = 2 - 2d$ .... (ii)

and $4c + d = 1$

or, $4(2 - 2d) + d = 1$ ($\because$ from equation (ii))

or, $8 - 8d + d = 1$

or, $-7d = 1 - 8$

$\therefore d = 1$

From (ii) $c = 2 - 2 \times 1 = 0$

Thus, the transformation matrix is (M) = $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

Example 8

The vertices of square ABCD are A(2, 3), B(4, 3), C(4, 5) and D(2, 5). When this square ABCD is transformed by a 2×2 matrix, the new vertices become A'(3, 2), B'(3, 4), C'(5, 4) and D'(5, 2), forming the square A'B'C'D'. Find the required 2×2 transformation matrix.

Solution: Here,

Let the 2×2 transformation matrix be = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Writing the vertices of square ABCD, A(2, 3), B(4, 3), C(4, 5) and D(2, 5) in matrix form Object (O) = $\begin{pmatrix} 2 & 4 & 4 & 2 \\ 3 & 3 & 5 & 5 \end{pmatrix}$

Writing the vertices of the image square A'B'C'D', A'(3, 2), B'(3, 4), C'(5, 4) and D'(5, 2), in matrix form:

Image (I) = $\begin{pmatrix} 3 & 3 & 5 & 5 \\ 2 & 4 & 4 & 2 \end{pmatrix}$

We know that, Image (I) = Transformation Matrix (TM) × Object (O)

or, $\begin{pmatrix} 3 & 3 & 5 & 5 \\ 2 & 4 & 4 & 2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} 2 & 4 & 4 & 2 \\ 3 & 3 & 5 & 5 \end{pmatrix}$

or, $\begin{pmatrix} 3 & 3 & 5 & 5 \\ 2 & 4 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 2a+3b & 4a+3b & 4a+5b & 2a+5b \\ 2c+3d & 4c+3d & 4c+5d & 2c+5d \end{pmatrix}$

Comparing corresponding elements:

$2a + 3b = 3$

or, $a = \frac{3-3b}{2}$..................(i)

$4a + 3b = 3$

or, $4\left(\frac{3-3b}{2}\right) + 3b = 3$ (∵ from equation (i))

or, $6 - 6b + 3b = 3$

or, $-3b = 3 - 6$

or, $-3b = -3$

Hence, $b = 1$

Substituting b = 1 to equation (i) $a = \frac{3-3\times1}{3} = 0$

Again, $2c + 3d = 2$

or, $c = \frac{2-3d}{2}$ .... (ii)

and $4c + 3d = 4$

or, $4\left(\frac{2-3d}{2}\right) + 3d = 4$ (∵ from equation (ii))

or, $4 - 6d + 3d = 4$

or, $-3d = 4 - 4$ $\therefore d = 0$

From eaution (ii) $c = \frac{2-3\times0}{2} = 1$

Hence, the required 2 × 2 transformation matrix is (M) = $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Example 9

Verify that: Rotation of +90° about the origin, followed by reflection in the line $y = x$ is equivalent to reflection in the X- axis.

Solution: Here,

The matrix representing rotation of 90° anticlockwise about the origin is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

The matrix representing reflection in the line $y = x$ is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

The matrix representing reflection in the X- axis is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Now, AB $= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 \times 0 + 1 \times 1 & 0 \times (-1) + 1 \times 0 \\ 1 \times 0 + 0 \times 1 & 1 \times (-1) + 0 \times 0 \end{pmatrix}$

$= \begin{pmatrix} 0 + 1 & 0 + 0 \\ 0 + 0 & -1 + 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ Hence, it is verified.

Exercise 9.3

1. **Tick (✓) the correct options for the given questions:**

A. Which transformation is represented by the given matrix $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$?

a. Reflection in the X- axis   b. Reflection in the Y- axis

c. Reflection in the line $y = x$ d. Reflection in the line $y = -x$

B. Which of the following matrices represents the unit square in the first quadrant?

a. $\begin{pmatrix} 0 & -1 & -1 & 0 \\ 0 & 1 & -1 & -1 \end{pmatrix}$ b. $\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix}$

c. $\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ d. $\begin{pmatrix} 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

C. Which is the 2 × 2 matrix related to reflection in the X-axis?

a. $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ b. $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

c. $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ d. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

D. Which is the 2 × 2 matrix related to a +90° rotation about the origin?

a. $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ b. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ c. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ d. $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

2. Define: Unit square and Matrix transformation

3. Transform the point A(–4, 6) using the given matrices.

a. $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ b. $\begin{pmatrix} 6 \\ 0 \end{pmatrix}$ c. $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ d. $\begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \end{pmatrix}$

4. a. Find the 2 × 2 matrix that transforms $(x, y) \rightarrow (2x + y, 3x + 4y)$.

b. Find the matrix that transforms $(x, y) \rightarrow (x - 2y, 2x - 3y)$.

5. a. The coordinates of PQ are P(3, 4) and Q(8, 4). Find the image coordinates after transformation using the given matrix $\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$.

b. A rectangle has vertices A(2, 3), B(2, 6), C(5, 2) and D(5, 6). After applying a 2 × 2 matrix $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, it becomes A'B'C'D'. Find the coordinates of A', B', C' and D'.

c. A unit square $\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ is transformed by a 2 × 2 matrix $\begin{pmatrix} 3 & 2 \\ 1 & 5 \end{pmatrix}$ into quadrilateral O'A'B'C'. Find the coordinates of O'A'B'C'.

6. a. Find a 2 × 2 matrix that transforms a unit square $\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ into a parallelogram $\begin{pmatrix} 0 & 3 & 4 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$.

b. Find a 2 × 2 matrix that transforms a unit square $\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ into a parallelogram $\begin{pmatrix} 0 & 3 & 4 & 1 \\ 0 & 1 & 3 & 2 \end{pmatrix}$.

c. Find a 2 × 2 matrix that transforms a unit square into a parallelogram $\begin{pmatrix} 0 & 2 & 5 & 3 \\ 0 & 3 & 4 & 1 \end{pmatrix}$.

7. a. Prove using matrix transformation that: A rotation of –90° about the origin followed by reflection in the line $x = 0$ is equivalent to reflection in the line $y = -x$.

b. Prove using matrix transformation that: A rotation of +90° about the origin followed by reflection in the Y- axis is equivalent to reflection in the line $y = x$.

Answer

1. A. d B. c C. a D. b 2. Show to the teacher.

3. a. (–1, 6) b. (2, 6), c. (–1, 8) d. $\left(-\frac{7}{2}, \frac{19}{3}\right)$ 4. a. $\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

b. $\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$ 5. a. P'(17, 7), Q'(32, 12) b. A'(2, 7), B'(2, 10), C'(5, 12) D'(5, 16)

c. O'(0, 0), A'(3, 1), B'(5, 6) C'(2, 5)

6. a. $\begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$ c. $\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}$

7. Show to the teacher.

Project Work

Take any triangular-shaped object and keep it on graph paper. Find coordinates of the vertices of that triangle. Transform the triangle in different ways using four 2×2 matrices. Present the objects and images of the triangles on the graph paper. Paste the object and image on same graph paper and present it in the classroom.

Miscellaneous Exercise – Within Content Area

1. A circle passes through the points (–2, 0) and (0, –2), and its center lies on the straight line $2x - 3y + 1 = 0$.
   a. Define the circle in terms of conic section.
   b. Find the coordinates of the center of circle.
   c. Find the equation of the line passing through the circle's center and perpendicular to the line $2x - 3y + 1 = 0$.
2. A line segment PQ passes through the points P(2, –3) and Q(–4, 9).
   a. Find the equation of the line PQ.
   b. Find the equation of a line parallel to PQ passing through the point M(–1, 1).
   c. Find the equation of a line perpendicular to PQ passing through the point N(1, 3).
   d. Using the translation vector PQ, translate points P(2, –3) and Q(–4, 9) and find the coordinates of the images P' and Q'.
3. A cone has a vertex angle of 70°. When the cone is intersected by a plane, a parabola is formed.
   a. Define the resulting conic in standard terms.
   b. Find the angle between the axis of the cone and the intersecting plane.

4. A circle has the equation $4x^2 + 4y^2 - 24x - 20y - 3 = 0$.
   a. If a plane intersects the cone parallel to its base, what type of conic is formed? Write it.
   b. Find the coordinates of the circle's center and its radius.
   c. If one end of a diameter of this circle is (1, 2), find the equation of the diameter.
5. A circle has the equation $x^2 + y^2 - 2x + 4y - 4 = 0$.
   a. Define the circle in standard form.
   b. Find the coordinates of the circle's center.
   c. If one end of a diameter is (1, 2) and a line passes through the diameter perpendicular to this end and through point B (–3, 1), find the equation of the line.
6. Let R_1 represent reflection about the X- axis and R_2 represent reflection about the Y- axis. The points A(1, 5), B(5, 1) and C(8, 3) are the vertices of triangle ABC.
   a. Find the equations of the sides AB and BC of triangle ABC.
   b. What single transformation does the composition $R_1 \circ R_2$ represent?
   c. Using this single transformation, find the image $\Delta A'B'C'$ of ΔABC.
   d. Draw ΔABC and its image $\Delta A'B'C'$ on the same diagram.
7. The points P(1, 2), Q(5, 2) and R(1, 5) are the vertices of triangle PQR.
   a. Write the transformation matrix representing reflection about the X- axis.
   b. Using the matrix $\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$, find the coordinates of the image of ΔPQR after the transformation.
   c. Find the equation of the perpendicular PS drawn from vertex P of ΔPQR to side QR.
8. Triangle MUN has vertices M(–2, 2), U(2, 2) and N(2, 8) and it is transformed by vectors $T_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $T_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.
   a. Find the composition of the transformations $T_1 \circ T_2$.
   b. Using the composite transformation $T_1 \circ T_2$, find the coordinates of the image $\Delta M'U'N'$ of ΔMUN.
   c. Draw ΔMUN and its image $\Delta M'U'N'$ on the same graph.

9. Let R_1 represent reflection about the line $y = x$, and R_2 represent reflection about the line $x = 0$. The vertices of triangle ABC are A(3, 2), B(1, 2), and C(2, 3).

   a. What single transformation does the composition $R_1 \circ R_2$ represent?

   b. Using this single transformation, find the image ΔA'B'C' of ΔABC.

   c. Draw ΔABC and its image ΔA'B'C' on the same graph.

   d. Find the equation of the altitude AD drawn from vertex A(3, 2) to side BC.

10. Triangle BUT has vertices B(3, 2), U(1, –1) and T(5, –5).

   a. If two straight lines have slopes m_1 and m_2 and the angle between them is θ, write the relationship between m_1, m_2 and θ.

   b. Find the equation of the line passing through the centroid G of ΔBUT and parallel to side UT.

   c. Using the matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, find the coordinates of the image ΔB'U'T' after transforming ΔBUT.

Answer

1. a. Show to the teacher. b. (1, 1) c. $3x + 2y - 5 = 0$
2. a. $2x + y - 1 = 0$ b. $2x + y + 1 = 0$ c. $x - 2y + 5 = 0$ d. P'(–4, 9), Q'(–10, 21)
3. a. Show to the teacher. b. 35°
4. a. Circle b. $(3, \frac{5}{2})$, 4 units c. (5, 3)
5. a. Show to the teacher b. (1,–2), 3 units c. $y - 1 = 0$
6. a. $x + y - 6 = 0$, $2x - 3y - 7 = 0$ b. 180° centre of rotation (0, 0) c. A'(−1, −5), B'(−5,−1) and C'(−8,−3) d. Show to the teacher.
7. a. show to the teacher. b. P'(7, 4) Q'(11, 12) R'(16, 7) c. $4x - 3y + 2 = 0$
8. a. $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ b. M'(2, 5), U'(6, 5) and N'(6, 11) c. Show to the teacher.
9. a. 270° centre of rotation (0,0) b. A'(2, –3), B'(2, –1) and C'(3, –2) c. Show to the teacher. d. $x + y - 5 = 0$
10. a. $\tan\theta = \pm\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$ b. $3x + 3y - 5 = 0$ c. B'(–3, –2), U'(–1, 1) and T'(–5, 5)

Lesson 10 Vector

10.1 Introduction

A physical quantity that represents both magnitude and direction is called a vector. The development of vectors, connecting it with modern algebra and geometry, is considered to have developed in the 19^{th} century. The modern analysis of vectors was done by William Rowan Hamilton (1805 – 1865) after he invented Quaternion in 1843. He developed the algebra of quaternions to represent magnitude and direction simultaneously. After that, American physicist Josiah Willard Gibbs and English mathematician Oliver Heaviside simplified Hamilton's algebra and developed vector analysis, which contributed significantly to the development of modern physics and mathematics. Its use is helpful in solving various problems in physics, mathematics, engineering, and computer science.

William Hamilton

10.2 Scalar Product

10.2.1 Introduction to Scalar Product

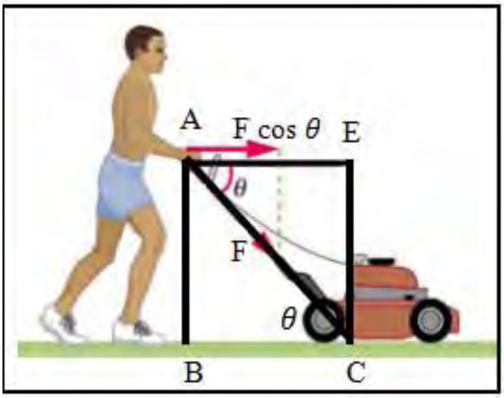
Activity 1

Observe the figure below and discuss the following questions:

a. Among the force, displacement, and work used by a person, which is a vector quantity and which is a scalar? Why?

b. How can the work done by the person be expressed in an equation?

c. How much work has that person done?



Here, a person is cutting grass using lawnmower by applying a force equal to FN and making an angle θ with the horizon. In this situation, the person has cut an amount of grass equal to d(BC). Therefore, the person applies a force of F.cosθ in the direction of displacement on the lawnmower. In this case, the force (F) and the distance moved (d) used by the person are both vector quantities because both represent direction and magnitude. However, work (W) is a scalar quantity because it only gives magnitude.

We know that,

$W = \vec{F}.\vec{d} = F\cos\theta d = |F|\,|d|\cos\theta$

Thus, $W = |F|\,|d|\cos\theta$

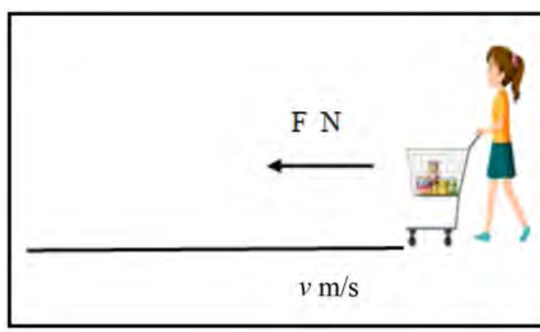
When a force is applied to any object, the product of the force in the direction of displacement and the distance moved is work which is a scalar quantity. This product of two vectors $\vec{F}$ and $\vec{d}$ is a scalar quantity because the result is a scalar. Therefore, this type of multiplication is called a scalar product. The unit of work is Joule.

When a force is applied to any object, the product of the force in the direction of displacement and the distance moved by the object is called work. Since work represents only magnitude, work is a scalar quantity. Work = Force × Displaced Distance, i.e., $W = \vec{F}.\vec{d} = |F|\,|d|\cos\theta$.

Activity 2

Observe the figure alongside and discuss the following questions:

A girl is pushing a trolley placed on a plain road with a force of F N. When she pushes the trolley in straight, the speed increases uniformly at a rate of V m/s in time t seconds.



a. Among the applied force and velocity, which are vector quantities and which are scalar quantities, state with reason.

b. Write the equation that shows the relationship between force and velocity.

c. How much power is applied by the girl to the trolley?

Here, the girl applies a force of F N. To move the trolley in a straight line, she applies a force of F cosθ in the direction of motion. In this situation, the force (F) applied by the girl and the velocity (v) of the trolley are both vector quantities because they both have direction and magnitude. However, the power used (P) is a scalar quantity because it has only magnitude.

We know that,

Power (P) = $\vec{F}.\vec{v}$ = F cosθ v = |F| |v|cosθ

Therefore, P = |F||v|cosθ

Thus, when force is applied to any object, the product of the force in the direction of displacement and the velocity of the object in the direction of that force is power which is scalar quantity. This product of two vectors $\vec{F}$ and $\vec{v}$ is a scalar quantity because the result is a scalar. Therefore, this multiplication is called the scalar product of vectors. The unit of power is watt.

When a force is applied to any object, the product of the force in the direction of displacement and the velocity of the object is called power. Since power represents only magnitude, power is a scalar quantity. Power = Force × Velocity, i.e., P = $\vec{F}$ $\vec{v}$ = |F| |v|cosθ

From the above two activities, it is clear that the scalar product of two vectors is defined as the product of the magnitude of the first vector, the magnitude of the second vector, and the cosine of the angle between them. Therefore, when two vectors are multiplied, the result is a scalar, which is why this concept is called the scalar product.

If two vectors $\vec{a}$ and $\vec{b}$ and the angle between them is θ(0° ≤ θ ≤ π), then, the scalar product of $\vec{a}$ and $\vec{b}$ is defined as $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$. The scalar product is also called the dot product. To find the angle θ between two vectors, the formula $\cos\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$ is used.

10.2.2 Conditions for Parallel and Perpendicular of Vectors

a. When two vectors are parallel to each other, the angle between them is 0° or 180°.

When θ = 0°, $\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$

or, $\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\cos 0°$

or, $\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\,.\,1$

Thus, $\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|$

When θ = 180°, $\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$

or, $\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\cos 180°$

or, $\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\,(-1)$

Thus, $\vec{a}.\vec{b} = -|\vec{a}|\,|\vec{b}|$

Thus, if vectors $\vec{a}$ and $\vec{b}$ are parallel, $\vec{a}.\vec{b} = \pm|\vec{a}|\,|\vec{b}|$.

b. When two vectors are perpendicular to each other, the angle between them is 90°.

$\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$

or, $\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\cos 90°$

Or, $\vec{a}.\vec{b} = |\vec{a}|\ |\vec{b}|\ .0$

Therefore, $\vec{a}.\vec{b} = 0$

Thus, if vectors $\vec{a}$ and $\vec{b}$ are perpendicular, $\vec{a}.\vec{b} = 0$.

10.2.3 Scalar Product in Terms of Coordinates

Let two vectors $\vec{a} = (x_1, y_1)$ and $\vec{b} = (x_2, y_2)$ Expressing vectors $\vec{a}$ and $\vec{b}$ in terms of unit vectors, $\vec{a} = x_1\vec{i} + y_1\vec{j}$ and $\vec{b} = x_2\vec{i} + y_2\vec{j}$

Now, $\vec{a}.\vec{b} = (x_1\vec{i} + y_1\vec{j}).(x_2\vec{i} + y_2\vec{j})$

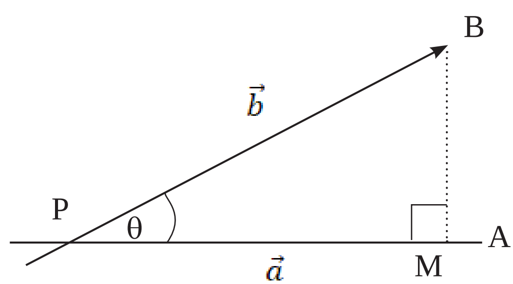
$= x_1\vec{i}.x_2\vec{i} + x_1\vec{i}.y_2\vec{j} + y_1\vec{j}.x_2\vec{i} + y_1\vec{j}.y_2\vec{j}$

$= x_1x_2 + y_1.y_2$ $\quad [\because \vec{i}.\vec{i} = 1 \text{ and } \vec{i}.\vec{j} = 0]$

For two vectors $\vec{a} = (x_1, y_1)$ and $\vec{b} = (x_2, y_2)$, $\vec{a}.\vec{b} = x_1x_2 + y_1.y_2$

10.2.4 Geometrical Meaning of Scalar Product

Let, $\overrightarrow{PA} = \vec{a}$ and $\overrightarrow{PB} = \vec{b}$ be two vectors, and the angle between them is θ ($0 \leq \theta \leq \pi$) Now, draw a perpendicular from point B to $\overrightarrow{PA}$, right-angled triangle PMB is formed.



In the triangle PMB,

$\cos\theta = \dfrac{PM}{PB} = \dfrac{PM}{|\vec{b}|}$

or, $\quad PM = |\vec{b}|\cos\theta$

Now, $\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$

= (magnitude of $\vec{a}$) (projection of $\vec{b}$ on $\vec{a}$) $\qquad PM = |\vec{b}|\cos\theta$

Similarly, $\vec{a}.\vec{b} = (|\vec{b}|)(|\vec{a}|\cos\theta)$

= (magnitude of $\vec{b}$) (projection of $\vec{a}$ and $\vec{b}$)

The geometrical meaning of the scalar product of two vectors is the product of the magnitude of the first vector and the projection of the second vector onto the first. It can also be considered as the product of the magnitude of the second vector and the projection of the first vector onto the second.

10.2.5 Properties of Scalar Product

a. Commutative property: If $\vec{a}$ and $\vec{b}$ are two vectors, then, $\vec{a}.\vec{b} = \vec{b}.\vec{a}$.

b. Distributive property: If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are vectors,

then, $\vec{a}.(\vec{b} + \vec{c}) = \vec{a}.\vec{b} + \vec{a}.\ \vec{c}$.

c. Multiplication by scalar: If $\vec{a}$ and $\vec{b}$ are two vectors and k is a scalar,

then, $(k\vec{a}).\vec{b} = \vec{a}.(k\vec{b}) = k(\vec{a}.\vec{b})$.

d. Maximum value: If two vectors are in the same direction $(\theta = 0°)$, $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|$. This is the maximum value of the scalar product.

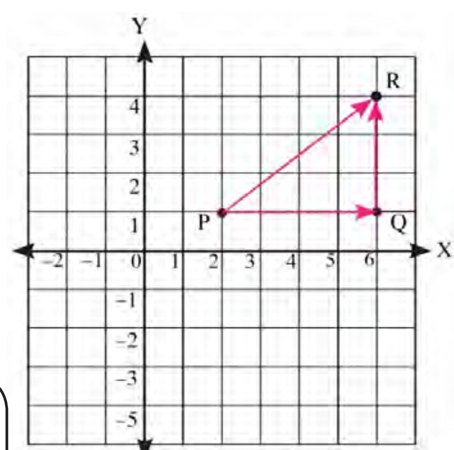
e. Minimum value: If two vectors are in opposite directions $(\theta = 180°)$, $\vec{a}.\vec{b} = -|\vec{a}||\vec{b}|$. This is the minimum value of the scalar product.

10.2.6 Triangle Law of Vector Addition

Study the given graph and write the answers to the following questions:

a. Express $\overrightarrow{PQ}$, $\overrightarrow{QR}$ and $\overrightarrow{PR}$ in terms of coordinates.

b. Find the relationship between $\overrightarrow{PQ}$, $\overrightarrow{QR}$ and $\overrightarrow{PR}$.



In the figure, $\overrightarrow{PQ}$ represents the displacement from point P to point Q, and $\overrightarrow{QR}$ represents the displacement from point Q to point R. Their combined displacement is represented by $\overrightarrow{PR}$ (displacement from point P to point R). Thus, $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$.

This rule of vector addition is called the triangle law of vector addition.

Though Provoking Question: How can $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ be represented in the form of two other vectors?

If two vectors are represented by two sides of a triangle in order, then, their sum is represented by the third side in the opposite direction.

Example 1

If $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$, find the value of $\overrightarrow{OA}.\overrightarrow{OB}$

Solution

Here, $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

We know that, $\overrightarrow{OA}.\overrightarrow{OB} = x_1x_2 + y_1y_2$

$= (2 \times 5) + (3 \times 1)$

$= 10 + 3 = 13$

Therefore, $\overrightarrow{OA}.\overrightarrow{OB} = 13$

Example 2

If $|\vec{a}|$ **= 5,** $|\vec{b}|$ **= 6 and the angle between them θ = 60°, find the value of** $\vec{a}.\vec{b}$

Solution

Here, $|\vec{a}| = 5$, $|\vec{b}| = 6$, $\theta = 60°$

Now, $\vec{a}.\vec{b} = |\vec{a}||\vec{b}| \cos\theta$

$= 5 \times 6 \cos 60°$

$= 5 \times 6 \times (\frac{1}{2}) = 15$

Therefore, $\vec{a}.\vec{b} = 15$

Example 3

The points A(1, -3) and B(3, 4) are given. Origin O(0, 0) is given. Check if $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are perpendicular to each other, verify your conclusion.

Solution

Here, $\overrightarrow{OA} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

We know, the scalar product $(\vec{a}.\vec{b}) = x_1x_2 + y_1y_2$

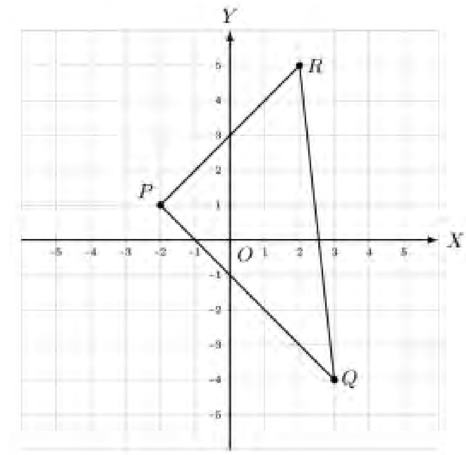
Here, $\overrightarrow{OA}.\overrightarrow{OB} = (1 \times 3) + (-3 \times 4)$

$= 3 - 12 = -9$

Therefore $\overrightarrow{OA}.\overrightarrow{OB} = -9$

We know, for two vectors to be perpendicular, their scalar product must be zero.

Thus, $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are not perpendicular.



Example 4

In the given figure, three points P(–2, 1), Q(3, –4), and R(2, 5) are given.

a. Find $\overrightarrow{PQ}.\overrightarrow{QR}$

b. If the value of $\overrightarrow{PQ}.\overrightarrow{QR}$ is negative, analyse the nature of ∠PQR (acute or obtuse).

Solution

Here, given points P(– 2, 1), Q(3, – 4) and R(2, 5)

a. If two points A(x_1, y_1) and B(x_2, y_2), then $\overrightarrow{AB} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$

Here, P(– 2, 1) and Q(3, – 4)

$$\overrightarrow{PQ} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix} = \begin{pmatrix} 3 - (-2) \\ -4 - 1 \end{pmatrix} = \begin{pmatrix} 3 + 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \end{pmatrix}$$

Similarly, Q(3, – 4) and R(2, 5)

$$\overrightarrow{QR} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix} = \begin{pmatrix} 2-3 \\ 5 - (-4) \end{pmatrix} = \begin{pmatrix} -1 \\ 5 + 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 9 \end{pmatrix}$$

Thus, $\overrightarrow{PQ} = \begin{pmatrix} 5 \\ -5 \end{pmatrix}$ and $\overrightarrow{QR} = \begin{pmatrix} -1 \\ 9 \end{pmatrix}$

$$\overrightarrow{PQ}.\overrightarrow{QR} = 5 \times (-1) + (-5) \times 9 = -5 - 45 = -50$$

Therefore, $\overrightarrow{PQ}.\overrightarrow{QR} = -50$

b. $\overrightarrow{PQ}.\overrightarrow{OR} = -50$ is negative. When the scalar product of two vectors is negative, the angle between them is obtuse (> 90°). Therefore, the interior angle ∠PQR of the triangle is acute.

Alternative method

To find ∠PQR, we need to see the scalar product of two vectors, $\overrightarrow{QP}$ and $\overrightarrow{QR}$ starting from point Q, i.e., $\overrightarrow{PQ} = -\overrightarrow{QP} = -\begin{pmatrix} 5 \\ 5 \end{pmatrix}$

$\overrightarrow{QR} = \begin{pmatrix} -1 \\ 9 \end{pmatrix}$

Here, $\overrightarrow{QP} . \overrightarrow{QR}$ = (– 5) × (– 1) + (5 × 9) = 5 + 45 = 50 (positive)

Since the scalar product is positive, the angle ∠PQR is acute.

Example 5

Two vectors $\overrightarrow{OA} = \begin{pmatrix} 5 \\ m \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ are given. The angle between them ∠AOB = 90°.

a. If two vectors are perpendicular, what is the value of their scalar product?

b. Find the value of *m* by finding $\overrightarrow{OA} . \overrightarrow{OB}$.

c. If $\overrightarrow{OA}$ and $\overrightarrow{OB}$ parallel instead of perpendicular, what could be the value of m? Compare them.

Solution

Here, $\overrightarrow{OA} = \begin{pmatrix} 5 \\ m \end{pmatrix}$ where $(x_1, y_1) = (5, m)$ and $\overrightarrow{OB} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ where $(x_2, y_2) = (4, -2)$ and the angle between them ∠AOB = 90°

a. If two vectors are perpendicular, their scalar product is always zero.

$\therefore \overrightarrow{OA} . \overrightarrow{OB} = 0$

b. Using the scalar product formula,

$\overrightarrow{OA} . \overrightarrow{OB} = x_1x_2 + y_1y_2$

or, $\overrightarrow{OA} . \overrightarrow{OB} = 5 \times 4 + m \times (-2)$

or, $\overrightarrow{OA} . \overrightarrow{OB} = 20 - 2m$

Since, ∠AOB = 90°: $\overrightarrow{OA} . \overrightarrow{OB} = 0$

$20 - 2m = 0$

or, $20 = 2m$

Therefore, m = 10

c. For vectors to be parallel,

If $\overrightarrow{OA}$ र $\overrightarrow{OB}$ then, $x_1y_2 = x_2y_1$

or, $5 \times (-2) = 4 \times m$

or, $-10 = 4m$

Thus, $m = -2.5$

For the two vectors to be perpendicular, m must be 10, but for the vectors them to be parallel, m must be –2.5. The values of m for these conditions are different.

Example 6

For three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are such that $|\vec{a}| = 3$, $|\vec{b}| = 5$, $|\vec{c}| = 4$ and $(\vec{a} + \vec{b} + \vec{c}) = (0, 0)$

a. Write the relationship between the square of a vector $\vec{a}$ $(\vec{a}^2)$ and the square of its magnitude $(|\vec{a}|)^2$.

b. Find the value of $\vec{a}.\vec{c}$.

c. Analyse the value of $\vec{a}.\vec{c}$ and the length of the sides, what type of triangle is these three vectors form?

Solution

Here, $\vec{a} + \vec{b} + \vec{c} = 0$, $|\vec{a}| = 3$, $|\vec{b}| = 5$ and $|\vec{c}| = 4$

a. The square of any vector is equal to the square of its magnitude, i.e., $(\vec{a})^2 = |\vec{a}|^2$.
b. From the above equation $(\vec{a} + \vec{c} = -\vec{b})$, squaring on both sides

$(\vec{a} + \vec{c})^2 = (-\vec{b})^2$

or, $(\vec{a})^2 + 2\vec{a}.\vec{c} + (\vec{c})^2 = (\vec{b})^2$

or, $|\vec{a}|^2 + 2\vec{a}.\vec{c} + |\vec{c}|^2 = |\vec{b}|^2$

or, $(3)^2 + 2\vec{a}.\vec{c} + (4)^2 = (5)^2$

or, $9 + 2\vec{a}.\vec{c} + 16 = 25$

or, $25 + 2\vec{a}.\vec{c} = 25$

or, $2\vec{a}.\vec{c} = 25 - 25$

or, $2\vec{a}.\vec{c} = 0$

$\therefore \vec{a}.\vec{c} = 0$

c. Since $\vec{a}.\vec{c} = 0$ these two vectors are perpendicular. The given three vectors form a right-angled triangle.

Exercise 10 .1

1. Tick (✓) the correct option for the given questions.

A. If angle between vectors $\vec{a}$ and $\vec{b}$ is θ, which one is scalar product of $\vec{a}$ and $\vec{b}$?

a. $\vec{a}.\vec{b} = |\vec{a}||\vec{b}| \sin\theta$ b. $\vec{a}.\vec{b} = |\vec{a}||\vec{b}| \cos\theta$

c. $\vec{a}.\vec{b} = |\vec{a}| + |\vec{b}| \cos\theta$ d. $\vec{a}.\vec{b} = |\vec{a}| + |\vec{b}| \sin\theta$

B . If $\vec{a} = (x_1, y_1)$ and $\vec{b} = (x_2, y_2)$, what is the value of $\vec{a}.\vec{b}$?

a. $x_1y_1 + x_2y_2$ b. $x_1x_2 - y_1y_2$

c. $x_1x_2 + y_1y_2$ d. $x_1y_2 + x_2y_1$

C. If $\vec{a}.\vec{b} = 0$, $|\vec{a}| \neq 0$ and $|\vec{b}| \neq 0$, what is the angle between $\vec{a}$ and $\vec{b}$?

a. 0° b. 45° c. 90° d. 180°

D. If $\vec{a}$ and $\vec{b}$ are parallel and in the same direction, what is the value of $\vec{a}.\vec{b}$?

a. $|\vec{a}||\vec{b}|$ b. $|\vec{a}||b|$ c. 0 d. 1

E. If $\vec{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$, what is the value of $\vec{a}.\vec{b}$?

a. 12 b. 0 c. –12 d. 6

F. If $\vec{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, what is the value of $\vec{a}^2$?

a. 5 b. –13 c. 13 d. 25

G. If $\overrightarrow{OA} = \begin{pmatrix} 5 \\ m \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ and $\angle AOB = 90°$, which is the value of m?

a. 10 b. –10 c. 6 d. –6

2. What is meant by the scalar product of two vectors? Explain with an example.
3. Under what conditions are two vectors perpendicular or parallel? Give reasons.
4. Find the scalar product of the following vectors:

a. $\vec{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ b. $\vec{a} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

c. $\vec{a} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ d. $\vec{p} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\vec{q} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

5. Find the angle between the following vectors:

a. $\vec{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ b. $\vec{a} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

c. $\vec{a} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ d. $\vec{p} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\vec{q} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

6. Determine whether the following vectors are perpendicular or parallel:

a. $\vec{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ b. $\vec{a} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

7. If the following vectors are perpendicular, find the value of m:

a. $\vec{i} = \begin{pmatrix} 2 \\ m \end{pmatrix}$ and $\vec{j} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ b. $\vec{a} = \begin{pmatrix} 3 \\ m \end{pmatrix}$ and $\vec{i} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$

c. $\vec{a} = 3\vec{i} + m\vec{j}$ and $\vec{b} = 6\vec{i} - 2\vec{j}$ d. $3\vec{i} - 2\vec{j}$ and $m\vec{i} + 3\vec{j}$

8. If the following vectors are parallel, find the value of m:

a. $\vec{i} = \begin{pmatrix} 2 \\ m \end{pmatrix}$ and $\vec{j} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ b. $\begin{pmatrix} 3 \\ m \end{pmatrix}$ and $\vec{i} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$

c. $\vec{a} = 3\vec{i} + m\vec{j}$ and $\vec{b} = 6\vec{i} - 2\vec{j}$ d. $3\vec{i} - 2\vec{j}$ and $m\vec{i} + 3\vec{j}$

9. $\vec{a} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ be two vectors.

a. What is the scalar product $\vec{a}.\vec{b}$?

b. If $\vec{a}.\vec{b} = 0$, what is the relationship between $\vec{a}$ and $\vec{b}$?

c. If $\vec{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$, find the value of $\vec{a}.\vec{b}$.

d. Under what condition is $\vec{a}.\vec{b}$ maximum? Discuss with reason.

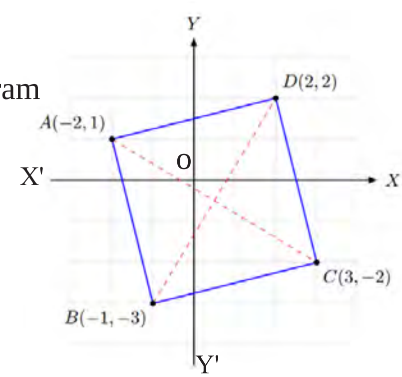
10. Four points A(–2, 1), B(–1, –3), C(3, –2) and D(2, 2) are given.

a. If A(x_1, y_1) and B(x_2, y_2), what is the vector $\overrightarrow{AB}$?

b. Express $\overrightarrow{AB}$ and $\overrightarrow{BD}$ in the form of column vectors.

c. Find the angle between $\overrightarrow{AC}$ and $\overrightarrow{BD}$.

d. Determine whether ABCD is a parallelogram or a rectangle based on the lengths of sides and angles.



11. Given that $\vec{a} + \vec{b} + \vec{c} = 0$

a. What does $\vec{a} + \vec{b} + \vec{c} = 0$ mean?

b. If $|\vec{a}| = 3$, $|\vec{b}| = 5$ and $|\vec{c}| = 4$, find $\vec{a}.\vec{c}$.

12. If $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$, prove that the vectors $\vec{a}$ and $\vec{b}$ are perpendicular to each other.

Answer

1\. A. b B. c C. c D. a E. b F. c G. a 2 - 3. Show to the teacher.

4\. a. 0 b. 6 c. 0 d. 0 5.a. 90° b. 0° c. 90° d. 90°

6\.a. Perpendicular b. Perpendicular 7.a. –3 b. $\frac{-9}{2}$

c. 9 d. 2 8. a. $\frac{4}{3}$ b. 2 c. –1 d. $\frac{-9}{2}$

9. a. $x_1x_2+y_1.y_2$ b. Perpendicular c. 23 d. Show to the teacher.

10. a. $\begin{pmatrix} x_2-x_1 \\ y_2-y_1 \end{pmatrix}$ b. $\begin{pmatrix} 1 \\ -4 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ c. 90° (d) Show to the teacher.

11. a. Show to the teacher. b. 0 12. Show to the teacher.

Project Work

Draw the X- axis and Y- axis on a graph paper and denote their point of intersection as the origin O(0, 0). Draw 5 small shapes or dots (in different quadrants) like house (A), tree (B), school (C), temple (D) and pond (E).

(a) Find the coordinates (x, y) of each object.

(b) Express these coordinates in the form of position vectors and column vectors.

(c) Based on the given table and questions, perform the analysis:

S.N.	Name of Objects	Symbol	Coordinates	Magnitude of Position Vector	Scalar Product
1.	House	A	....	....	AO and OB
2.	....	....	....	....	OA and OB
3.	....	....	....	....	AB and BC
4.	....	....	....	....	....
5.	....	....	....	....	....

Prepare a report on this and present it in the class.

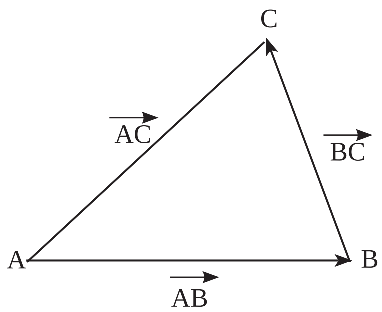
10.3 Vector Geometry

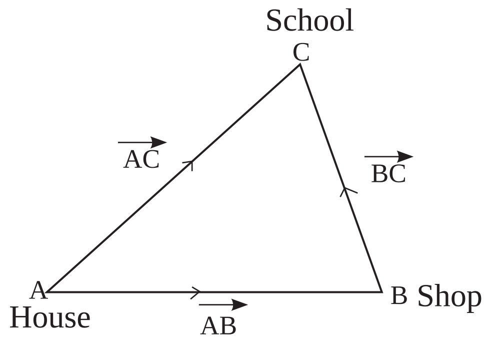
Activity 1

Suppose you need to go from your home (point A) to school (point C). What are the possible paths in this situation? You can go directly from point A to point C in a straight line. Another path is from point A to point B, and then, from point B to point C.

a. If you go from point A to point B and then, to point C, is the total distance equal to, less than, or greater than the straight-line distance AC? Discuss.

b. Considering the concept of distance and direction, you went from A to B and then from B to C. Considering the path taken, does $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ Discuss.





10.3.1 Mid-point formula

In any triangle, the median drawn from a vertex is half the sum of the vectors representing the two sides originating from that vertex.

Proof

Let, D, E and F be midpoints of sides BC, AC and AB respectively in ΔABC. Here, AD is a median of triangle ABC.

In triangle ABD, using the triangle law of vector addition,

$\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD}$ .... (i)

Similarly, in triangle ΔACD

$\overrightarrow{AD} = \overrightarrow{AC} + \overrightarrow{CD}$ .... (ii)

Adding equations (i) and (ii),

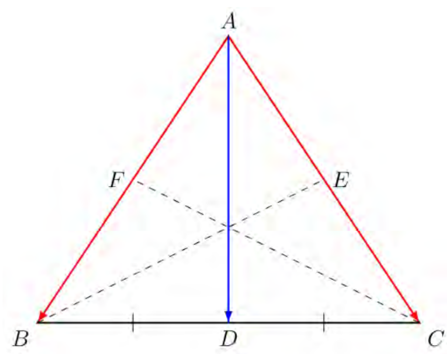
$\overrightarrow{AD} + \overrightarrow{AD} = (\overrightarrow{AB} + \overrightarrow{AC}) + (\overrightarrow{BD} + \overrightarrow{CD})$

or, $2\overrightarrow{AD} = (\overrightarrow{AB} + \overrightarrow{AC}) + (\overrightarrow{BD} + \overrightarrow{CD})$

or, $2\overrightarrow{AD} = (\overrightarrow{AB} + \overrightarrow{AC}) + \overrightarrow{0}$

or, $2\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{AC}$

$\therefore \overrightarrow{AD} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC})$



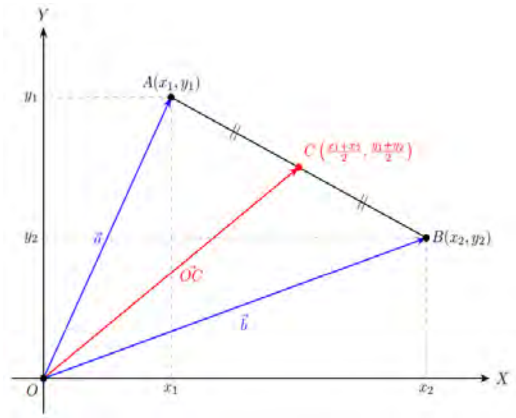
Similarly, the following relations can be established for other medians.

$\overrightarrow{BE} = \frac{1}{2}(\overrightarrow{BA} + \overrightarrow{BC})$ and $\overrightarrow{CF} = \frac{1}{2}(\overrightarrow{CA} + \overrightarrow{CB})$

If the coordinates of the endpoints are given, the midpoint's position vector can be found.

In the given figure, point A has coordinates (x_1, y_1) and point B has coordinates (x_2, y_2). The position vector of the midpoint C of line segment AB is

$$\overrightarrow{OC} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$



10.3.2 Internal DivisionTheorem

If the position vectors of points A and B are $\vec{a}$ and $\vec{b}$ respectively and point P divides the line segment AB internally in the ratio $m{:}n$, then, the position vector $\vec{p}$ is given by

$$\vec{p} = \frac{n\vec{a} + m\vec{b}}{m + n}$$

Proof

Let O be the origin. $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OP} = \vec{p}$

Point P divides AB internally in the ratio $m : n$, i.e., AP : PB = $m : n$.

or, $\frac{\text{AP}}{\text{PB}} = \frac{m}{n}$

$\therefore n\overrightarrow{AP} = m\,\overrightarrow{PB}$ ....(i)

We know that, $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$

$= \vec{p} - \vec{a}$ and $\overrightarrow{PB} = \overrightarrow{OB} - \overrightarrow{OP} = \vec{b} - \vec{p}$

Substituting these values in equation (i):

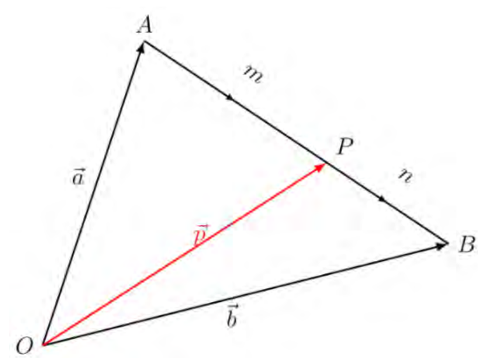
$n(\vec{p} - \vec{a}) = m(\vec{b} - \vec{p})$

or, $n\vec{p} - n\vec{a} = m\vec{b} - m\vec{p}$

or, $n\vec{p} + m\vec{p} = n\vec{a} + m\vec{b}$

or, $(n + m)\vec{b} = n\vec{a} + m\vec{b}$

$\therefore \vec{p} = \frac{n\vec{a} + m\vec{b}}{m + n}$



Therefore, the position vector of point which divides line segment internally is $\vec{p} = \dfrac{n\vec{a} + m\vec{b}}{m+n}$

10.3.3 External Division Theorem

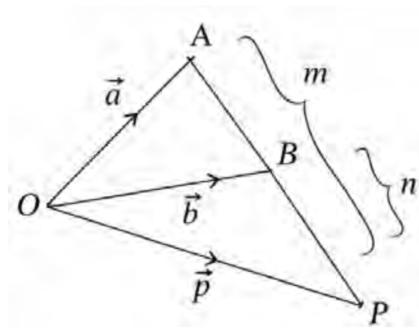
If the position vectors of points A and B are $\vec{a}$ and $\vec{b}$ respectively and point P divides the line segment AB externally in the ratio m:n then, the position vector $\vec{p}$ is given by,

$$\overrightarrow{OP} = \vec{p} = \frac{m\vec{b} - n\vec{a}}{m-n}$$

Proof

Let O be the origin. $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ Point P divides AB externally in the ratio m:n, i.e., AP : PB = m : n.

Points A, B and P are collinear.



$$\frac{AP}{BP} = \frac{m}{n}$$

$\therefore n\,\overrightarrow{AP} = m\,\overrightarrow{BP}$ ....(i)

We know, $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \vec{p} - \vec{a}$ and $\overrightarrow{BP}$

$= \overrightarrow{OP} - \overrightarrow{OB} = \vec{p} - \vec{b}$

Substituting in equation (i):

n $(\vec{p} - \vec{a}) = m(\vec{p} - \vec{b})$

or, $n\vec{p} - n\vec{a} = m\vec{p} - m\vec{b}$

or, $m\vec{b} - n\vec{a} = m\vec{p} - n\vec{p}$

or, $m\vec{b} - n\vec{a} = (m-n)\,\vec{p}$

$\therefore \quad \vec{p} = \dfrac{m\vec{b} - n\vec{a}}{m-n}$

Therefore, position vector of a point which divides the line segment $\vec{p} = \dfrac{m\vec{b} - n\vec{a}}{m-n}$

10.4 Theorems

As we proved geometric theorems in plane geometry, we can also prove some theorems of triangles using vector addition and scalar product.

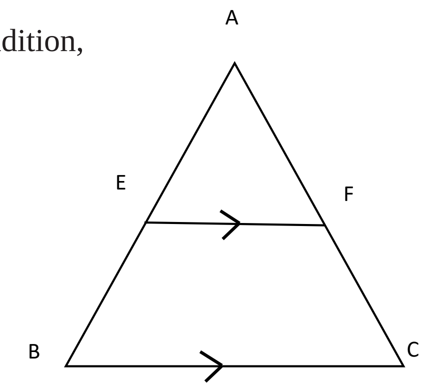
Theorem 1: Mid-point Theorem

The line segment joining the midpoints of two sides of a triangle is parallel to the third side and half of it.

Proof

Given: In triangle ABC, points E and F are the midpoints of sides AB and AC respectively.

To prove: $\overrightarrow{EF} = \frac{1}{2}\overrightarrow{BC}$ and EF//BC



In triangle AEF, by the triangle law of vector addition,

$\overrightarrow{EF} = \overrightarrow{EA} + \overrightarrow{AF}$ .... (i)

$\overrightarrow{EA} = \frac{1}{2}\overrightarrow{BA}$ and $\overrightarrow{AF} = \frac{1}{2}\overrightarrow{AC}$ [∵ E and F are midpoints of AB and AC respectively.]

Now, from equation (i),

$\overrightarrow{EF} = \frac{1}{2}\overrightarrow{BA} + \frac{1}{2}\overrightarrow{AC}$

or, $\overrightarrow{EF} = \frac{1}{2}(\overrightarrow{BA} + \overrightarrow{AC})$

or, $\overrightarrow{EF} = \frac{1}{2}\overrightarrow{BC}$ (∵ In ∆ABC: $\overrightarrow{BA} + \overrightarrow{AC} = \overrightarrow{BC}$)

$\therefore \overrightarrow{EF} = \frac{1}{2}\overrightarrow{BC}$

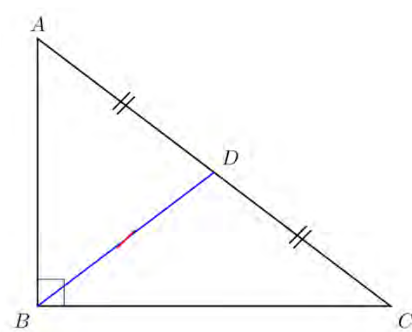
According to the definition of parallel vectors $\overrightarrow{EF} = k.\overrightarrow{BC}$ where $k = \frac{1}{2}$, vectors $\overrightarrow{EF}$ and $\overrightarrow{BC}$ are parallel.

Thus, the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half of it.

Theorem 2

In a right-angled triangle, the midpoint of the hypotenuse is equidistant from all three vertices.

Proof



Given: Triangle ABC is a right-angled triangle, where ∠B = 90°. AC is the hypotenuse. Point D is the midpoint of AC.

To prove: The point D is equidistance from the vertices A, B and C i.e.AD = BD = CD.

Here, ∠ABC = 90°, so $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are perpendicular.

So, $\overrightarrow{BA}.\overrightarrow{BC} = 0$ .... (i)

In triangle ABD, $\overrightarrow{BA} = \overrightarrow{BD} + \overrightarrow{DA}$ and in the triangle BCD, $\overrightarrow{BC} = \overrightarrow{BD} + \overrightarrow{DC}$

Substituting in equation (i): $\overrightarrow{BA}.\overrightarrow{BC} = 0$

or, $(\overrightarrow{BD} + \overrightarrow{DA}).(\overrightarrow{BD} + \overrightarrow{DC}) = 0$

or, $(\overrightarrow{BD} + \overrightarrow{DA}).(\overrightarrow{BD} - \overrightarrow{DA}) = 0$ ($\because$ D is mid point of AC then, $\overrightarrow{DC} = -\overrightarrow{DA}$.

or, $(\overrightarrow{BD})^2 - (\overrightarrow{DA})^2 = 0$ [$\because (\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$

or, $|\overrightarrow{BD}|^2 - |\overrightarrow{DA}|^2 = 0$

or, $BD^2 = DA^2$

or, BD = DA

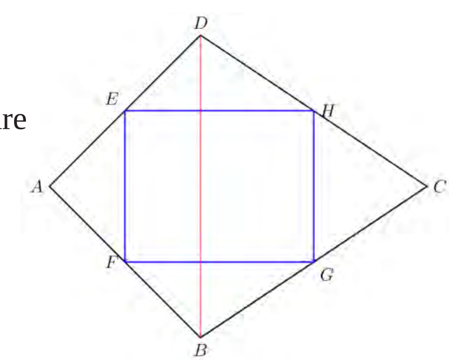
Here, D is the midpoint, so DA = DC.

Thus, AD = DC = BD

Hence, the midpoint of the hypotenuse of a right-angled triangle is equidistant from all three vertices.

Theorem 3: Mid points of sides of quadrilateral

The quadrilateral formed by joining the midpoints of the sides of any quadrilateral is a parallelogram.



Proof

Given: A quadrilateral ABCD. F, G, H, and E are the midpoints of sides AB, BC, CD, and AD respectively. Joining these midpoints sequentially forms quadrilateral EFGH.

To prove: EFGH is a parallelogram,

$\overrightarrow{EF} = \overrightarrow{HG}$ or $\overrightarrow{EH} = \overrightarrow{FG}$)

Construction: Join vertices B and D.

In triangle ABD, F and E are the midpoints of AB and AD respectively. By the midpoint theorem, the line joining the midpoints of two sides is parallel to the third side and half of it.

$\overrightarrow{EF} = \frac{1}{2}\overrightarrow{DB}$ .... (i)

Similarly, in triangle ΔCBD,

$\overrightarrow{HG} = \frac{1}{2}\overrightarrow{DB}$ ....(ii)

From equations (i) and (ii), $\overrightarrow{EF} = \overrightarrow{HG}$

Equal vectors are always parallel, so EF// HG

Similarly, by joining vertices A and C, we can prove $\overrightarrow{EH} = \overrightarrow{FG}$ and $\overrightarrow{EH} // \overrightarrow{FG}$.

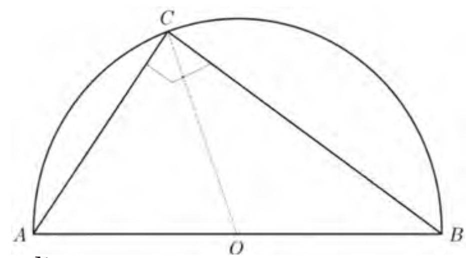
Therefore, EFGH is a parallelogram.

Hence, the quadrilateral formed by joining the midpoints of the sides of any quadrilateral is a parallelogram.

Theorem 4

The angle subtended by the diameter of a circle is a right angle.

Proof



Given: O is the center of a circle. AB is the diameter. C is any point on the circumference.

To prove: ∠ACB is a right angle (90°).

The distance from the center of the circle to any point on the circumference is equal to the radius.

Thus, $|\overrightarrow{OA}| = |\overrightarrow{OB}| = |\overrightarrow{OC}| = r$

Now, $\overrightarrow{AC}.\overrightarrow{BC} = (\overrightarrow{OC} - \overrightarrow{OA}).(\overrightarrow{OC} - \overrightarrow{OB})$

$= (\overrightarrow{OC} - \overrightarrow{OA}).(\overrightarrow{OC} - \overrightarrow{AO})$ $[\overrightarrow{AO} = \overrightarrow{OB}]$

$= (\overrightarrow{OC} - \overrightarrow{OA}). (\overrightarrow{OC} + \overrightarrow{OA})$ $[\overrightarrow{OA} = -\overrightarrow{AO}]$

$= (\overrightarrow{OC})^2 - (\overrightarrow{OA})^2$

$= |\overrightarrow{OC}|^2 - |\overrightarrow{OA}|^2$

$= OA^2 - OA^2 [\because\ |\overrightarrow{OC}| = |\overrightarrow{OA}| = r]$

$= 0$

Since $\overrightarrow{AC}.\overrightarrow{BC} = 0$, $\overrightarrow{AC}$ and $\overrightarrow{BC}$ are perpendicular.

Hence, the angle subtended by the diameter of a circle is a right angle.

Example 1

If the position vectors of points A and B are $4\vec{i} + 3\vec{j}$ and $2\vec{i} - \vec{j}$ respectively, find the position vector of the midpoint M of AB.

Solution

Here, $\overrightarrow{OA} = 4\vec{i} + 3\vec{j}$ and $\overrightarrow{OB} = 2\vec{i} - \vec{j}$

By the midpoint theorem,

$$\overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB}) = \frac{1}{2}(4\vec{i} + 3\vec{j} + 2\vec{i} - \vec{j}) = \frac{1}{2}(6\vec{i} + 2\vec{j}) = 3\vec{i} + \vec{j}$$

Thus, the position vector of M is $3\vec{i} + \vec{j}$.

Example 2

The position vectors of points A and B are $x_1\vec{i} + y_1\vec{j}$ and $x_2\vec{i} + y_2\vec{j}$. Point P divides the line segment AB internally in the ratio *m*:*n*. Express the position vector of P in terms of $\vec{i}$ and $\vec{j}$.

Solution

Here, $\overrightarrow{OA} = \vec{a} = x_1\vec{i} + y_1\vec{j}$ and $\overrightarrow{OB} = \vec{b} = x_2\vec{i} + y_2\vec{j}$

Point P divides AB internally in the ratio m:n.

Using the internal division formula,

$$\overrightarrow{OP} = \frac{m\overrightarrow{OB} + n\overrightarrow{OA}}{m+n}$$

$$= \frac{n(x_1\vec{i} + y_1\vec{j}) + m(x_2\vec{i} + y_2\vec{j})}{m+n}$$

$$= \frac{nx_1\vec{i} + ny_1\vec{j} + mx_2\vec{i} + my_2\vec{j}}{m+n}$$

$$= \frac{(mx_2 + nx_1)\vec{i} + (my_2 + ny_1)\vec{j}}{m+n}$$

Therefore, $\overrightarrow{OP} = \dfrac{(mx_2 + nx_1)\vec{i} + (my_2 + ny_1)\vec{j}}{m+n}$

Example 3

The position vectors of points A and B are $\overrightarrow{OA} = 4\vec{i} + 3\vec{j}$ and $\overrightarrow{OB} = 3\vec{i} - 4\vec{j}$

a. If point M divides AB internally in the ratio 3:2, find the position vector of M.

b. If point N divides AB externally in the ratio 5:3, find the position vector of N.

Solution

Here, $\overrightarrow{OA} = 4\vec{i} + 3\vec{j}$ and $\overrightarrow{OB} = 3\vec{i} - 4\vec{j}$

a. For internal division point M, *m*:*n* = 3:2.

Using the internal division formula,

$$\overrightarrow{OM} = \frac{n\overrightarrow{OA} + m\overrightarrow{OB}}{m+n}$$

or, $$\overrightarrow{OM} = \frac{2(4\vec{i} + 3\vec{j}) + 3(3\vec{i} - 4\vec{j})}{3+2}$$

or, $$\overrightarrow{OM} = \frac{(8\vec{i} + 6\vec{j}) + (9\vec{i} - 12\vec{j})}{5}$$

or, $$\overrightarrow{OM} = \frac{(8+9)\vec{i} + (6-12)\vec{j}}{5}$$

or, $$\overrightarrow{OM} = \frac{17\vec{i} - 6\vec{j}}{5}$$

or, $$\overrightarrow{OM} = \frac{17}{5}\vec{i} - \frac{6}{5}\vec{j}$$

Therefore, the value of $\overrightarrow{OM}$ is $\frac{17}{5}\vec{i} - \frac{6}{5}\vec{j}$.

b. Here, $\overrightarrow{OA} = \vec{a} = 4\vec{i} + 3\vec{j}$ and $\overrightarrow{OB} = \vec{b} = 3\vec{i} - 4\vec{j}$. For external division point N, $m:n = 5:3$.

Using the external division formula,

$$\overrightarrow{ON} = \frac{m\vec{b} - n\vec{a}}{m-n}$$

or, $$\overrightarrow{ON} = \frac{5(3\vec{i} - 4\vec{j}) - 3(4\vec{i} + 3\vec{j})}{5-3}$$

or, $$\overrightarrow{ON} = \frac{(15\vec{i} - 20\vec{j}) - (12\vec{i} + 9\vec{j})}{2}$$

or, $$\overrightarrow{ON} = \frac{(15-12)\vec{i} + (-20-9)\vec{j}}{2}$$

or, $$\overrightarrow{ON} = \frac{3}{2}\vec{i} - \frac{29}{2}\vec{j}$$

Therefore, the value of $\overrightarrow{ON}$ is $\frac{3}{2}\vec{i} - \frac{29}{2}\vec{j}$.

Exercsie 10.2

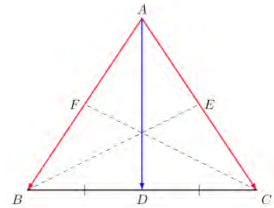
1. Tick (✓) the correct option for the given questions:

A. In triangle ABC, what is $\overrightarrow{AB} + \overrightarrow{BC}$ equal to?

a. $\overrightarrow{CA}$ b. $\overrightarrow{BA}$

c. $\overrightarrow{AC}$ d. $\overrightarrow{CB}$

B. In the given triangle ABC, D, E, F are midpoints. What is $2\overrightarrow{AD}$ equal to?



a. $\overrightarrow{AB} - \overrightarrow{AC}$ b. $\overrightarrow{AB} + \overrightarrow{AC}$

c. $\overrightarrow{AC} - \overrightarrow{AB}$ d. $\frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC})$

C. If the position vectors of A and B are $\vec{a}$ and $\vec{b}$, what is the position vector of the midpoint of AB?

a. $\vec{a} + \vec{b}$ b. $\frac{\vec{a} - \vec{b}}{2}$ c. $\frac{\vec{a} + \vec{b}}{2}$ d. $2(\vec{a} + \vec{b})$

D. If $\overrightarrow{OA} = 4\vec{i} + 3\vec{j}$ and $\overrightarrow{OB} = 3\vec{i} - 4\vec{j}$ what is the position vector of the midpoint of AB?

a. $7\vec{i} - \vec{j}$ b. $\frac{1}{2}(7\vec{i} - \vec{j})$ c. $\vec{i} + 7\vec{j}$ d. $\frac{1}{2}(\vec{i} + 7\vec{j})$

E. If point P divides the line segment joining points with position vectors $\vec{a}$ and $\vec{b}$ internally in the ratio m:n, what is the position vector $\vec{p}$?

a. $\frac{m\vec{a} + n\vec{b}}{m + n}$ b. $\frac{n\vec{a} + m\vec{b}}{m + n}$ c. $\frac{n\vec{a} - m\vec{b}}{n - m}$ d. $\frac{m\vec{b} + n\vec{a}}{m - n}$

2. a. If the position vectors of points P and Q are $4\vec{i} + 7\vec{j}$ and $4\vec{i} - 3\vec{j}$, find the position vector of the midpoint R of PQ.

b. If the position vectors of points R and S are $5\vec{i} - 4\vec{j}$ and $3\vec{i} + 6\vec{j}$, find the position vector of the midpoint T of RS.

3. Points A(2, –3) and B(3, 2) are endpoints of line segment AB.

a. If M divides AB internally in the ratio 2:3, find the position vector of M.

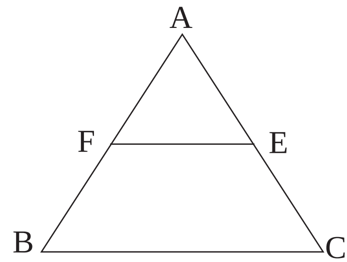
b. If N divides AB externally in the ratio 3:1, find the position vector of N.

4. The position vectors of two points A and B are $\overrightarrow{OA} = 6\vec{i} - 2\vec{j}$ and $\overrightarrow{OB} = 2\vec{i} + 3\vec{j}$.

   a. If P divides AB internally in the ratio 3:1, find the position vector of P.

   b. If Q divides AB externally in the ratio 2:1, find the position vector of Q.

5. In triangle ABC, F and E are the midpoints of sides AB and AC respectively.



   a. State the mid-point theorem in vector form.

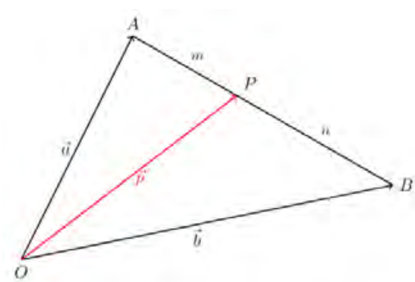
   b. Express $\overrightarrow{BC}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ using the law of triangle addition.

   c. Prove using vector method that $\overrightarrow{EF} = \frac{1}{2}\overrightarrow{CB}$ (or, $\overrightarrow{FE} = \frac{1}{2}\overrightarrow{BC}$).

   d. If D is the midpoint of BC, prove that $\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = 0$.

6. The position vectors of points A and B are $\vec{a}$ and $\vec{b}$. Point P divides AB internally in the ratio m:n.

   a. Write the formula to find the position vector $\vec{p}$ of the internal division point P.



   b. In the figure, why is $n\overrightarrow{AP} = m\overrightarrow{PB}$ used when AP:PB = m:n? Give reason.

   c. If $\vec{a} = \vec{i} + 2\vec{j}$ and $\vec{b} = 4\vec{i} + 8\vec{j}$, find the position vector of the point that divides AB internally in the ratio 1:2.

   d. Under what condition does $\vec{p} = \frac{\vec{a} + \vec{b}}{2}$ hold? Give reason.

7. Point P divides the line segment joining points with position vectors $\vec{a}$ and $\vec{b}$ externally in the ratio m:n.

   a. Write the formula to find the position vector $\vec{p}$ for external division.

   b. Prove using vector method that $\vec{p} = \frac{m\vec{b} - n\vec{a}}{m - n}$

   c. If $\vec{a} = \vec{i} + 4\vec{j}$ and $\vec{b} = 5\vec{i} - 3\vec{j}$, find the position vector of the point that divides AB externally in the ratio 1:2.

   d. If m:n = 1:1, does the external division point P exist or not? Discuss with reason.

8. **Prove by using vector method:**

   a. The quadrilateral formed by joining the midpoints of the sides of any quadrilateral is a parallelogram.

   b. The angle subtended by the diameter of a circle is a right angle.

Answer

1. A. c B. b C. c D. b E. b 2. a. $4\vec{i} + 2\vec{j}$ b. $4\vec{i} + 2\vec{j}$

3. a. $\frac{12}{5}\vec{i} - \vec{j}$ b. $\frac{7}{2}\vec{i} + \frac{19}{2}\vec{j}$ 4. a. $3\vec{i} + \frac{7}{4}\vec{j}$ b. $-2\vec{i} + 8\vec{j}$

5– 8. Show to the teacher.

Miscellaneous Exercise – Within Content Area

1. Two vectors $\vec{a} = 3\vec{i} + 4\vec{j}$ and $\vec{b} = x\vec{i} - 2\vec{j}$

   a. If $\vec{a} = (x_1, y_1)$ and $\vec{b} = (x_2, y_2)$ express $\vec{a}.\vec{b}$ in terms of $\vec{i}$ and $\vec{j}$

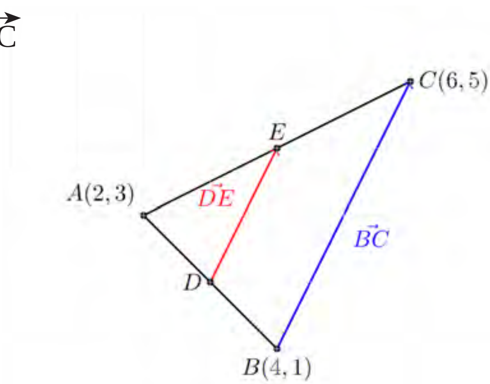
   b. If $\vec{a}$ and $\vec{b}$ are perpendicular, find the value of x.

   c. For what value of x does vector $\vec{b}$ make an angle 45° with the Y-axis? Find.

   d. If the position vectors of points A and B are $2\vec{i} + 4\vec{j}$ and $4\vec{i} - 2\vec{j}$, find the position vector of the midpoint of AB.

2. In triangle ABC, the position vectors of points are

   $\overrightarrow{OA} = 2\vec{i} + 3\vec{j}$, $\overrightarrow{OB} = 4\vec{i} + \vec{j}$ and $\overrightarrow{OC}$ $= 6\vec{i} + 5\vec{j}$



a. If D is the midpoint of A and B, express $\overrightarrow{OD}$ in terms of $\vec{i}$ and $\vec{j}$.

b. If E is the midpoint of A and C, express $\overrightarrow{OE}$ in terms of $\vec{i}$ and $\vec{j}$.

c. Using vector method, prove that $\overrightarrow{DE} = \frac{1}{2}\overrightarrow{BC}$.

d. Are $\overrightarrow{DE}$ and $\overrightarrow{BC}$ parallel? Verify.

3. The position vectors of points P and Q are $\vec{p} = 2\vec{i} + \vec{j}$ and $\vec{q} = \vec{i} - 3\vec{j}$. Point R divides PQ in the ratio 3:2.

   a. Write the formula to find the position vector of a point dividing the line joining two points in the ratio *m*:*n*.

   b. Find the position vector $\vec{r}$ of point R.

   c. Find the value of ∠POR.

   d. Compare ∠POR and ∠QOR.

4. The position vectors of points A and B are $3\vec{i} + 2\vec{j}$ and $\vec{i} - 4\vec{j}$. Point P divides AB externally in the ratio 2:3.

   a. Write the formula to find the position vector of a point dividing the line joining two points externally in the ratio *m*:*n*.

   b. Find the position vector of point P.

   c. If $m > n$, on which side of A or B does the external division point P lie? Explain with reason.

   d. Find the value of ∠AOB.

Answer

1. a. $x_1 x_2 + y_1 y_2$ b. $\frac{8}{3}$ c. ±2 d. $3\vec{i} + \vec{j}$

2. a. $\overrightarrow{OD} = 3\vec{i} + 2\vec{j}$; $\overrightarrow{OE} = 4\vec{i} + 4\vec{j}$ b – d. Show to the teacher.

3. a. $\frac{n\overrightarrow{OA} + m\overrightarrow{OB}}{m + n}$ b. $\frac{7}{5}(\vec{i} - \vec{j})$ c. $\cos^{-1}\frac{1}{\sqrt{10}}$ d. Show to the teacher.

4. a. $\frac{n\vec{b} - m\vec{a}}{m - n}$ b. $7\vec{i} + 14\vec{j}$ c. Show to the teacher. d. $\cos^{-1}\frac{-5}{\sqrt{221}}$

Project Work

Draw X-axis and Y-axis on a large graph paper. Draw a map of a city on this graph. Mark four main places of that city and write their coordinates on the map.

a. School (A): at some coordinates (e.g., $2\vec{i} + 5\vec{j}$)

b. Hospital (B): (e.g., $6\vec{i} + 2\vec{j}$)

c. Cinema Hall (C): (e.g., $3\vec{i} + 4\vec{i}$)

d. Playground (D): (e.g., $2\vec{i} - 3\vec{j}$)

e. Determine where to place a 'Library' between any two places and mark it.

f. Check whether the two paths are perpendicular (90°) or not.

Lesson 11

Statistics

11.1. Introduction

The branch of mathematics which deals with the collection, analysis, interpretation, and presentation of data is called statistics. The history of statistics is as old as human civilization itself. In the early period, statistics was mainly used for the administration of the state. In ancient times, statistics was used in civilizations such as Babylonia, Egypt, and China to count population and resources. John Graunt (1620 AD -1674 AD) was the first person to present data scientifically by analyzing the mortality records of London. In the twentieth century, Sir Ronald A. Fisher laid the foundation of modern statistics. Therefore, he is known as the Father of Statistics. Later, Karl Pearson and William Sealy Gosset (1876 AD - 1937 AD) made further important contributions to this field. In today's digital age, computers and big data have made statistics more powerful and widespread. Statistics is now used in science, technology, and almost every field of study.

Dispersion

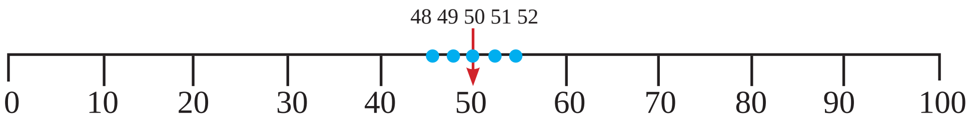
In statistics, whenever we study a set of data, we usually begin by finding the measure of central tendency, such as the mean, median, or mode. These measures represent the central or average position of the data. Measures of central tendency help us to understand more information in a simple and meaningful way.

However, measures of central tendency alone do not always give a complete information of the data. Even if two or more data sets have the same central value, their distribution or spread of values may be very different. For example, consider the marks obtained by students of three different groups:

Marks obtained by group A	48	49	50	51	52	50
Marks obtained by group B	40	45	50	55	60	50
Marks obtained by group C	20	35	50	65	80	50

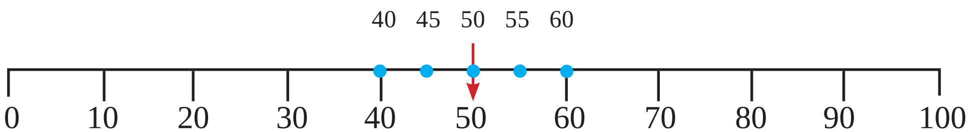
Data A: Low variation

mean = 50



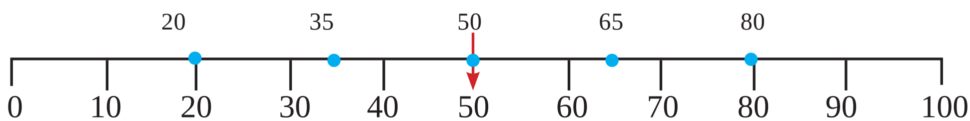
Data B: Moderate variation

mean = 50



Data C : High variation

mean = 50



As shown above, the mean of all three data sets is 50, but their nature is different:

a. Data set A (low variation):

Here, all the marks are very close to the mean (50). This shows that the performance of the students is almost similar.

b. Data set B (moderate variation):

Here, the marks are spread out at a moderate distance from the mean.

c. Data set C (high variation):

Here, the marks are widely spread out. This shows a large difference in the performance of the students.

If we consider only the mean, we may wrongly conclude that the performance of students in all three groups is the same. However, the main factor that makes these data sets different is their spread or degree of variation.

> The term dispersion (or variation) refers to how far the observations of a data set differ from one another or from the central value. The measure of dispersion tells us how much the data values are spread out. It helps us understand the extent to which the observations deviate from the mean or from each other.

Measures of dispersion help us in the following ways:

a. To determine whether the data are consistent or variable

b. To compare the reliability of two or more data sets.

c. To check how well the measure of central tendency represents the data.

d. To gain deeper insight for decision-making in fields such as education, business, economics, and science.

Thus, while measures of central tendency describe the centre of the data, measures of dispersion explain the spread of the data. To understand any data set completely, both measures are essential.

11.2 Quartile Deviation

Activity 1

Discuss the following questions:

a. Measure the height (in centimeters) of all the students in your class and write them on the board.

b. Present the collected height data in a table.

c. Find the mean, median, first quartile (Q_1), third quartile (Q_3), quartile deviation, and coefficient of quartile deviation from the data.

d. Can the spread of the data from the central value be measured by calculating range, quartile deviation, mean deviation, standard deviation and it's coefficient?

Activity 2

The weights (in kg) of 40 students in a class are given below. Study the given data.

40	40.5	45	47	60	61	56.4	39.5	56	52
42	62.5	69.5	42	43	46	55	56.5	35	51
58	61	49	57	51	59	47	52	57	59.5
48.5	58	54	67	46	56	39	48	51	60

a. Discuss how the above data can be presented in a table.

b. If we write each weight separately, what kind of table will be formed?

c. Let us divide the above data into class interval of 5.

Weight (In kg)	Tally bars	Number of students
35 - 40	||	2
40 - 45	~~||||~~ |	6
45 - 50	...	8
50 - 55	...	6
55 - 60	...	11
60 - 65	...	5
65 - 70	...	2

For example, in the interval 55–60 kg, the weights include all values from 55 kg up to 60 kg, such as 55, 56, 56.5, 57, 58, 59, 59.5, but not 60 which belongs to the next interval 60–65.

The data which can be expressed not only in whole numbers but also in decimals, and is spread over a definite interval, is called a continuous series.

11.3 Quartile Deviation of Continuous Series

Quartile Deviation / Semi Interquartile Range and its Coefficient

Activity 3

In a mathematics examination of a class, the marks obtained by the students are given in the table below. Discuss in groups and find: (a) first quartile,(b) third quartile,(c) quartile deviation, and (d) coefficient of quartile deviation.

Marks obtained	10 - 20	20 - 30	30 - 40	40 - 50	50 - 60	60 - 70
Number of students	2	5	10	11	5	2

Thought Provoking Question: Can the value of the coefficient of quartile deviation be greater than 1?

The value that divides the given data into 25% below and 75% above is called the first quartile (Q_1). The value that divides the data into 50% below and 50% above is called the median or second quartile (Q_2). Similarly, the value that divides the data into 75% below and 25% above is called the third quartile (Q_3).

The difference between the third (upper) quartile and the first (lower) quartile is called the interquartile range. "Interquartile Range" = $Q_3 - Q_1$

Half of the difference between the third quartile and the first quartile is called the quartile deviation. "Quartile Deviation (Q.D.)" = $\frac{Q_3 - Q_1}{2}$ · The coefficient of quartile deviation is given by "Coefficient of Q.D." = $\frac{Q_3 - Q_1}{Q_3 + Q_1}$ ·

Example 1

From the given data, calculate the quartile deviation and the coefficient of quartile deviation.

Wages (in hundred rupees)	6 - 8	8 - 10	10 - 12	12 - 14	14 - 16	16 - 18
Number of workers	60	75	85	70	50	10

Solution

Here, calculation of quartile deviation and its coefficient:

Wages (in hundred rupees)	Frequency (f)	Cumulative frequency (cf)
6 - 8	60	60
8 - 10	75	135
10 - 12	85	220
12 - 14	70	290
14 - 16	50	340
16 - 18	10	350
	N = 350	

Now, the position of $Q_1 = \left(\frac{N}{4}\right)^{th}$ item $= \left(\frac{350}{4}\right)^{th}$ item $= 87.5^{th}$ item

From the cumulative frequency table, the 87.5^{th} item lies in the class interval 8 - 10.
Therefore, the Q_1 class = 8 – 10

We have, $Q_1 = l + \frac{\frac{N}{4} - cf}{f} \times i$.....................(1)

Where, $l = 8$

$f = 75$

$cf = 60$

$i = 10 - 8 = 2$

Now, from (i):

$$Q_1 = 8 + \frac{87.5 - 60}{75} \times 2$$

$$= 8 + 0.73 \quad = 8.73$$

The position of $Q_3 = \left(\frac{3N}{4}\right)^{th}$ item $= \left(\frac{3 \times 350}{4}\right)^{th}$ item $= (262.5)^{th}$ item

From the cumulative frequency table, the 262.5[th] item lies in the class interval 12 - 14.
Therefore, the Q_3 class = 12 - 14.

We have, $Q_3 = l + \dfrac{\frac{3N}{4} - cf}{f} \times i$

$$Q_3 = 12 + \frac{262.5 - 220}{70} \times 2$$

$$= 12 + 1.21 = 13.21$$

Quartile deviation (Q.D.) $= \dfrac{Q_3 - Q_1}{2}$

$$= \frac{13.21 - 8.73}{2} = \frac{4.48}{2} = 2.24$$

Coefficient of Q.D. $= \dfrac{Q_3 - Q_1}{Q_3 + Q_1}$

$$= \frac{13.21 - 8.73}{13.21 + 8.73}$$

$$= \frac{4.48}{21.94} = 0.20$$

Therefore, quartile deviation = 2.24 and coefficient of quartile deviation = 0.20

Example 2

From the given data, calculate the quartile deviation and the coefficient of quartile deviation.

Working time (hours)	Less than 2	Less than 4	Less than 6	Less than 8	Less than 10	Less than 12
Number of workers	5	7	8	15	25	30

Solution

Working time (hours)	No. of workers (*f*)	Cumulative frequency (*cf*)
Less than 2	5	5
2 - 4	7 – 5 = 2	7
4 - 6	8 – 7 =1	8
6 - 8	15 – 8 = 7	15
8 - 10	25 – 15 = 10	25
10 - 12	30 – 25 = 5	30
	N = 30	

First quartile (Q_1) $= \left(\frac{N}{4}\right)^{th}$ item $= \left(\frac{30}{4}\right)^{th}$ item $= (7.5)^{th}$ item

From the cumulative frequency table, the 7.5th item lies in the class interval 4 - 6.
Therefore, Q_1 class = 4 - 6

$$Q_1 = l + \frac{\frac{N}{4} - cf}{f} \times i$$

$$Q_1 = 4 + \frac{7.5 - 7}{1} \times 2 = 4 + 1 = 5$$

Third quartile (Q_3) $= \left(\frac{3N}{4}\right)^{th}$ item $= \left(\frac{3 \times 30}{4}\right)^{th}$ item $(22.5)^{th}$ item

From the cumulative frequency table, the 22.5th item lies in the class interval 8 – 10.

Therefore, Q_3 class = 8 – 10

$$Q_3 = l + \frac{\frac{3N}{4} - cf}{f} \times i$$

$$Q_3 = 8 + \frac{22.5 - 15}{10} \times 2 = 8 + 1.5 = 9.5$$

Quartile Deviation (Q.D.) $= \frac{Q_3 - Q_1}{2} = \frac{9.5 - 5}{2} = \frac{4.5}{2} = 2.25$

Coefficient of Q.D. $= \frac{Q_3 - Q_1}{Q_3 + Q_1}$

$= \frac{9.5 - 5}{9.5 + 5} = \frac{4.5}{14.5} = 0.31$

Finally, Quartile Deviation (Q.D.) = 2.25 and Coefficient of Quartile Deviation = 0.31

Exercise 11.1

1. **Tick (✓) the correct option for the given questions:**

A. What is another term for quartile deviation?

a. Range b. Mean deviation
c. Standard deviation d. Semi–interquartile range

B. What percentage of the data does quartile deviation include in a distribution?

a. 25 %, b. 50 %, c. 75 %, d. 100 %

C. Which formula is used to find Quartile Deviation (Q.D.)?

a. $Q_3 - Q_1$ b. $\frac{Q_3 - Q_1}{2}$ c. $\frac{Q_3 + Q_1}{2}$ d. $\frac{Q_3 - Q_1}{Q_3 + Q_1}$

D. If $Q_1 = 10$ and $Q_3 = 30$, what is the value of the quartile deviation?

a. 10 b. 20 c. 30 d. 40

E. Which values are ignored while calculating quartile deviation ?

a. All the values of the data b. The middle 50% values

c. The first 25% and the last 25% values d. The average value

F. If quartile deviation is 10 and the upper quartile is 30, what is the value of the lower quartile?

a. 40 b. 30 c. 20 d. 10

2. From the given data, calculate the quartile deviation and the coefficient of quartile deviation.

a.

Obtained marks	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50
No. of students	4	10	20	10	6

b.

Wages (Rs. in thousands)	0 - 5	5 - 10	10 - 15	15 - 20	20 - 25	25 - 30
No. of workers	4	6	3	8	12	7

c.

Weight (Kg)	20 - 30	30 - 40	40 - 50	50 - 60	60 - 70	70 - 80	80 - 90
No. of students	4	5	10	15	8	7	1

d.

Obtained marks	$0 \le x < 10$	$10 \le x < 20$	$20 \le x < 30$	$30 \le x < 40$	$40 \le x < 50$
No. of students	5	10	15	10	5

e.

Obtained marks (less than)	<20	<30	<40	<50	<60
No. of students	5	12	27	40	50

f.

Wages (more than) per hour (Rs.)	100	200	300	400	500
No. of workers	100	80	60	40	20

3. **The marks obtained in Mathematics by 30 students of a class are given below:**

12	25	36	45	18	22	33	54	42	29
38	41	15	20	48	52	30	24	19	37
44	21	35	40	14	26	49	31	23	50

a. Taking the first-class interval as 10–20, present the given data in a frequency distribution table.

b. Find the value of the first quartile.

c. Find the value of the third quartile.

d. Calculate the quartile deviation.

e. Find the coefficient of quartile deviation.

Answer

1. A. d B. b C. b D. a E. b F. c

2. a. 7.5, 0.29 b. 6.88, 0.41 c. 10.44, 0.19 d. 8.75, 0.35 e. 8.88, 0.23

f. 125, 0.36 3. a. Show to the teacher. b. 22.63 c. 43.07 d. 10.22 e. 0.31

11.4 Mean Deviation

Activity 1

In a school, the marks obtained by students in a Mathematics examination of full marks 75 are given below:

Marks	10 - 20	20 - 30	30 - 40	40 - 50	50 - 60	60 - 70
Number of students	5	7	8	10	6	4

a. Based on the given data, find the required values and complete the following table:

Obtained marks	No. of students (f)	Mid value (m)	fm	$D = m - \overline{X}$	$\lvert D \rvert = \lvert m - \bar{X} \rvert$	$f\lvert D \rvert$
10 - 20	5					
20 - 30	7					
30 - 40	8					
40 - 50	10					
50 - 60	6					
60 - 70	4					
	N =		$\Sigma fm =$		$\Sigma f\lvert D \rvert =$	

b. Find the mean deviation from mean.

c. Find the coefficient of mean deviation from mean.

d. Based on the given data, find the required values and complete the following table:

Obtained marks	No. of students	*cf*	Mid value (*m*)	$\lvert D\rvert = \lvert m - M_d\rvert$	$f\lvert D\rvert$
10 - 20	5				
20 - 30	7				
30 - 40	8				
40 - 50	10				
50 - 60	6				
60 - 70	4				
	N =			$\Sigma f\lvert D\rvert =$	

e Find the mean deviation from median.

f. Find the coefficient of mean deviation from median.

The mean deviation of a dataset measures how far each value is from the central value (mean, median, or mode). It shows the spread of the data. When calculating mean deviation, the difference between each value and the central value is considered positive, regardless of whether the actual difference is positive or negative. This is done by taking the absolute value of the difference.

The mean deviation of a dataset is the average of the absolute differences between each data value and the mean or median value.

Formulas for Mean and Median Deviation

1. Mean deviation about the mean (MD) $= \dfrac{\Sigma f\,\lvert m - \overline{X}\rvert}{N} = \dfrac{\Sigma f\lvert D\rvert}{N}$

2. Coefficient of mean deviation from mean $= \dfrac{\text{Mean deviation from mean}}{\text{Mean}} = \dfrac{\text{MD}}{\text{Mean}}$

3. Mean deviation about the median (MD)= $= \dfrac{\Sigma f\,\lvert m - \text{Md}\rvert}{N} = \dfrac{\Sigma f\lvert D\rvert}{N}$

4. Coefficient of mean deviation from median = $\dfrac{\text{Mean deviation from median}}{\text{Median}} = \dfrac{\text{MD}}{\text{Median}}$

Example 1

A school has 40 students in class 10, whose weights range from more than 35 kg up to less than 70 kg. The data is given in the table below. Find the mean deviation from the mean and its coefficient for this dataset.

Weight (kg)	Number of Students
More than 35	40
More than 40	36
More than 45	30
More than 50	20
More than 55	15
More than 60	10
More than 65	4

Solution: Here,

From the table,

Weight	No. of students (f)	Mid value (m)	fm	$\|D\| = \|m - \bar{X}\|$	$f\|D\|$
35 - 40	40 – 36 = 4	37.5	150	14.375	57.5
40 - 45	36 – 30 = 6	42.5	255	9.375	56.25
45 - 50	30 – 20 = 10	47.5	475	4.375	43.75
50 - 55	20 – 15 = 5	52.5	262.5	0.625	3.125
55 - 60	15 – 10 = 5	57.5	287.5	5.625	28.123
60 - 65	10 – 4 = 6	62.5	375	10.625	63.75
65 - 70	4	67.5	270	15.625	62.5
	N = 40		$\Sigma fm = 2075$	$\Sigma f\|D\| = 315$	

Mean$(\bar{X}) = \frac{\Sigma fm}{N} = \frac{2075}{40} = 51.875$

Mean deviation from mean(MD)$= \frac{\Sigma f|D|}{N} = \frac{315}{40} = 7.875$

Coefficient of mean deviation from mean $= \frac{MD}{\bar{X}} = \frac{7.875}{51.875} = 0.1518$

Hence, mean deviation from mean = 7.875 and coefficient of MD from mean= 0.1518.

Exercise 11.2

1. **Tick (✓) the correct option for the given questions:**

A. What is the formula to calculate Mean Deviation (MD)?

a. $\frac{\Sigma fD}{N}$ b. $\frac{MD}{M_d}$ c. $\frac{MD}{Mean}$ d. $\frac{\Sigma f|D|}{N}$

B. If all the values of a data set are equal, what will be the mean deviation?

a. 1 b. Infinity c. 0 d. can't say

C. Which measure of central tendency can be used to calculate mean deviation?

a. Mean b. Median c. Mode d. All of the above

D. If $\Sigma f|m - \bar{X}| = 510$, $N = 30$ and mean = 40, what will be the coefficient of mean deviation?

a. 12.75 b. 17 c. 0.425 d. 0.75

2. **Calculate the mean deviation and its coefficient from mean of the given data.**

a.

Weight (kg)	0 - 4	4 - 8	8 - 12	12 - 16	16 - 20	20 - 24
Number of children	7	7	10	15	7	6

b.

Obtained marks	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50	50 - 60
No. of students	8	12	20	40	12	8

3. **Calculate the mean deviation and its coefficient from median of the given data.**

a.

Weight (kg)	20 - 30	30 - 40	40 - 50	50 - 60	60 - 70	70 - 80
No. of students	7	12	18	28	16	14

b.

Weight (kg)	3 - 5	5 - 7	7 - 9	9 - 11	11 - 13
No. of dogs	1	4	6	4	1

4. **A school conducted a mathematics exam of 50 full marks, and the scores obtained by 10 students are given.**

Obtained marks	Less than 10	Less than 20	Less than 30	Less than 40	Less than 50
No. of students	1	3	7	9	10

a. Find the mean deviation from the mean and its coefficient from the given data.

b. Find the mean deviation from the median and its coefficient from the given data.

5. In a factory, the workers' hourly wages range from Rs. 100 to less than Rs. 800. The data is given in the table. Find the mean deviation from the mean and its coefficient for this data.

Per hour wages (Rs)	No. of workers
More than 100	60
More than 200	56
More than 300	50
More than 400	40
More than 500	20
More than 600	10
More than 700	4

6. In a school, the marks obtained in the Mathematics by 28 students of a class 10 are given below.

15	62	50	70	74	34	60
41	14	48	28	38	64	21
57	40	34	30	47	20	75
45	56	29	22	13	53	66

a. Present the above data in a frequency table with the first class interval as 10 – 20.

b. Find the mean deviation from the mean and its coefficient.

c. Find the mean deviation from the median and its coefficient.

Answer

1. A. d B. c C. d D. c

2. a. 5.08, 0.42 b. 10.40, 0.34 3. a. 11.53, 0.21 b. 1.5, 0.1875

4. a. 8, 0.32 b. 8, 0.32 5. 113.3, 0.25 6. a. Show to the teacher.

b. 15.81, 0.35 c. 15.85, 0.36

Project Work:

Collect the marks obtained by your classmates in two unit tests of Mathematics.

a. Present the collected marks in a frequency table using appropriate class intervals.

b. Calculate the average (mean) marks.

c. Compare the two sets of marks based on the average marks.

d. Determine the quartile deviation.

e. Find the mean deviation and its coefficient.

f. How would you compare the two sets of marks based on the mean deviation?

11.4 Standard Deviation

Activity 1

In a certain examination, the marks obtained by two students are given below. Study these marks and answer the following questions.

Student A: 68, 70, 72, 69, 71, 70, 70

Student B: 40, 95, 50, 90, 60, 70, 80

a. Which student has the better result?

b. How can we determine which student's result is better?

c. Find the mean marks of student A and student B.

d. If both students have the same mean marks, how can their results be compared? Find it.

e. Find the standard deviation of the marks of student A and student B.

The mean marks of both student A and student B are 70. However, the nature of their marks is not the same. student A has obtained almost similar marks in all subjects, whereas student B has obtained very high marks in some subjects and very low marks in others.

The standard deviation of student A is 1.19 and the standard deviation of student B is 19.45. If the mean is the same but the standard deviations are different, what conclusion can be drawn?

Any measure that indicates how far or how close the data values are spread from their mean is called a measure of dispersion known as Standard Deviation. Among the different measures of dispersion (Range, Quartile Deviation, Mean Deviation, and Standard Deviation), Standard Deviation is the most widely used and considered the most reliable measure. Standard deviation is generally represented by the Greek letter σ (small sigma).

The square of the standard deviation is called variance, and it is written symbolically as σ^2.

Thought Provoking Question: Why is standard deviation considered the most appropriate measure of dispersion?

11.4.1 Uses of Standard Deviation

Standard deviation is an important measure of dispersion and is used in many fields, such as:

a. **Education:** To compare consistency or variation in students' examination results

b. **Weather Forecasting:** To measure the variation in average temperature or rainfall

c. **Stock Market:** To determine the risk involved in a particular stock or investment

d. **Quality Control:** To check whether the products manufactured in a factory have uniform quality or not

11.4.2 Coefficient of Standard Deviation

The value obtained by dividing the standard deviation by the mean is called the Coefficient of Standard Deviation. The coefficient of standard deviation is useful for comparing data sets that have different units. The smaller the coefficient of standard deviation, the greater the uniformity or stability in the distribution of data. Similarly, the larger the coefficient of standard deviation, the less the uniformity or stability and the greater the variability in the distribution of data. For this purpose, the following formula is used: Coefficient of Standard Deviation = $\frac{\sigma}{\bar{X}}$.

where, σ = Standard Deviation, $\bar{x}$ = Mean

11.5 Coefficient of Variation

The percentage form of the coefficient of standard deviation is called the Coefficient of Variation (C.V.). Since the coefficient of standard deviation lies between 0 and 1, it is usually converted into percentage form for practical use. It is denoted by C.V. As in the coefficient of standard deviation, the smaller the coefficient of variation, the greater the uniformity or stability in the data distribution. Similarly, the larger the coefficient of variation, the less the uniformity or stability and the greater the variability in the data distribution.

For this purpose, the following formula is used:

Coefficient of Variation (C.V.) $= \frac{\sigma}{\bar{X}} \times 100\%$

Standard Deviation of Continuous Series

There are different methods for finding the standard deviation of a continuous series.

Actual Mean Method

In this method, the standard deviation is calculated by finding the difference between actual mean and mid-value of the observation.
For this purpose, the following formula is used:

standard deviation $(\sigma) = \sqrt{\frac{\Sigma f(m-\bar{X})^2}{N}}$ .... (i)

Where: m = mid-value, N = sum of frequencies, $\bar{X}$ = mean

Steps for Calculating Standard Deviation

a. Find the mid-value (m) of each class interval.

b. Multiply each mid-value (m) by its corresponding frequency (f) and find fm. Then find the sum of fm (Σfm).

c. Calculate the mean ($\bar{x}$).

d. Find the deviation of each class mark from the mean ($m - \bar{x}$) and then find its square $(m - \bar{x})^2$.

e. Multiply each squared deviation by the corresponding frequency to obtain $f(m - \bar{x})^2$.

f. Find the sum of $f(m - \bar{x})^2$.

g. Substitute the values into the formula above (i) to calculate the standard deviation.

Example 1

From the given data, calculate the standard deviation and the coefficient of standard deviation by using the actual mean method.

Salary (In thousands)	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50	50 - 60
No. of employees	8	12	20	40	12	8

Solution

Salary	No. of employees (f)	Mid-value (m)	fm	$(m-\overline{X})^2$ $(m-31)^2$	$f(m-\overline{X})^2$
0 - 10	8	5	40	676	5408
10 - 20	12	15	180	256	3072
20 - 30	20	25	500	36	720
30 - 40	40	35	1400	16	640
40 - 50	12	45	540	196	2352
50 - 60	8	55	440	576	4608
	N = 100		$\Sigma fm =$ 3100		$\Sigma fm^2 =$ 16800

Mean $(\overline{X}) = \frac{\Sigma fm}{N} = \frac{3100}{100} = 31$

Standard deviation $(\sigma) = \sqrt{\frac{\Sigma f(m-\overline{X})^2}{N}} = \sqrt{\frac{16800}{100}} = \sqrt{168} = 12.96$

Hence, standard deviation $(\sigma) = 12.96$

Coefficient of standard deviation $= \frac{\sigma}{\overline{X}} = \frac{12.96}{31} = 0.418$

Hence, standard deviation $(\sigma) =$ 12.96 and it's coefficient = 0.418

Direct Method

The method of finding standard deviation directly from the data without using the actual mean or an assumed mean is called the direct method. In this method, the values in the data are used directly to calculate the standard deviation. The following formula or relation is used for this purpose. This method is suitable when the numbers in the data are small.

Standard deviation $(\sigma) = \sqrt{\frac{\Sigma fm^2}{N} - \left(\frac{\Sigma fm}{N}\right)^2}$ ……………. (ii)

Where, m = mid-value, N = sum of frequencies

Steps

a. Find the mid-value (m).

b. Multiply each m by its corresponding frequency (f) and obtain fm.

c. Multiply each m^2 by its corresponding frequency (f) and obtain fm^2.

d. Find the sum of fm (Σfm) and the sum of fm^2 (Σfm^2).

e. Substitute the values into the above formula (ii) to calculate the standard deviation.

Example 2

Calculate the standard deviation of the given data by using the direct method.

Obtained marks	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50
No. of students	3	5	7	3	2

Solution

Obtained marks	No. of students (f)	Mid value (m)	fm	fm^2
0 - 10	3	5	15	75
10 - 20	5	15	75	1125
20 - 30	7	25	175	4375
30 - 40	3	35	105	3675
40 - 50	2	45	90	4050
	N = 20		$\Sigma fm = 460$	$\Sigma fm^2 = 13300$

$$\text{Standard deviation } (\sigma) = \sqrt{\frac{\Sigma fm^2}{N} - \left(\frac{\Sigma fm}{N}\right)^2}$$

$$= \sqrt{\frac{13300}{20} - \left(\frac{460}{20}\right)^2} = \sqrt{665 - 529} = 11.66$$

Hence, standard deviation $(\sigma) = 11.66$

Short-cut Method or Assumed Mean Method

In this method, the standard deviation is calculated by using the deviation of the class marks from an assumed mean.For this purpose, the given formula or relation is used. This method is especially useful when the actual mean results in decimal values, making calculations more complicated.

$$\text{Standard deviation } (\sigma) = \sqrt{\frac{\Sigma fd^2}{N} - \left(\frac{\Sigma fd}{N}\right)^2} \ldots\ldots\ldots\ldots\ldots (iii)$$

Where, m = mid-value, N = sum of frequencies, $d = m - A$, A = assumed mean

Steps to Find Standard Deviation by Short-cut (Assumed Mean) Method

a. Find the mid-value (m).

b. Choose a suitable class mark as the assumed mean (A) and find the deviation $d = x - A$.

c. Multiply each d by its corresponding frequency (f) to obtain fd.

d. Multiply each d^2 by its corresponding frequency (f) to obtain fd^2.

e. Find the sum of fd (Σfd) and the sum of fd^2 (Σfd^2).

f. Substitute the values into the above formula (iii) to calculate the standard deviation.

Example 3

Calculate the standard deviation of the given data by using the short-cut (assumed mean) method.

Obtained marks	20 - 30	30 - 40	40 - 50	50 - 60	60 - 70
No. of students	10	15	20	8	7

Solution

Obtained marks	No. of students (f)	Mid-value (m)	$d = (m - A) = (m - 45)$	fd	fd^2
20 - 30	10	25	–20	–200	4000
30 - 40	15	35	–10	–150	1500
40 - 50	20	45 (A)	0	0	0
50 - 60	8	55	10	80	800
60 - 70	7	65	20	140	2800
	N = 60			$\Sigma fd = -130$	$\Sigma fd^2 = 9100$

Standard deviation $(\sigma) = \sqrt{\frac{\Sigma fd^2}{N} - \left(\frac{\Sigma fd}{N}\right)^2} = \sqrt{\frac{9100}{60} - \left(\frac{-130}{60}\right)^2}$

$= \sqrt{151.67 - 4.696} = \sqrt{146.974} = 12.12$

Hence, standard deviation $(\sigma) = 12.12$

Step Deviation Method

When the numbers used in the data are large and the calculation of the actual mean becomes difficult, the step deviation method makes the calculation of standard deviation easier. For this purpose, the following formula or relation is used.

$$\text{Standard deviation } (\sigma) = \sqrt{\frac{\Sigma f d'^2}{N} - \left(\frac{\Sigma f d'}{N}\right)^2} \times h \ldots\ldots\ldots\ldots (iv)$$

m = mid-value, N = sum of frequencies, $d' = \frac{d}{h} = \frac{m-A}{h}$, h = class interval,
A = assumed mean

Steps

a. Find the mid-value (m).

b. Choose a suitable mid-value as the assumed mean (A). Find: $d' = \frac{x-A}{h}$ by dividing the deviation of each class mark from A by the class interval (h).

c. Multiply each d' by its corresponding frequency (f) to obtain fd'.

d. Multiply each $(d')^2$ by its corresponding frequency (f) to obtain $f(d')^2$.

e. Find the sum of fd' i.e. $\Sigma fd'$ and the sum of $f(d')^2$ i.e. $\Sigma f(d')^2$.

f. Substitute the values into the above formula (iv) to calculate the standard deviation.

Example 4

Find the standard deviation of the given data using the step deviation method.

Wages Rs. (per hour)	0 - 100	100 - 200	200 - 300	300 - 400	400 - 500
No. of workers	5	8	7	3	2

Solution

Wages Rs. (Per hour)	No. of workers (f)	Mid-value (m)	$d' = \frac{m-250}{100}$	fd'	fd'^2
0 - 100	5	50	–2	–10	20
100 - 200	8	150	–1	–8	8
200 - 300	7	250	0	0	0
300 - 400	3	350	1	3	3
400 - 500	2	450	2	4	8
	N = 25			$\Sigma fd' = -11$	$\Sigma fd'^2 = 39$

Standard deviation $(\sigma) = \sqrt{\frac{\Sigma fd'^2}{N} - \left(\frac{\Sigma fd'}{N}\right)^2} \times h$

$$= \sqrt{\frac{39}{25} - \left(\frac{-11}{25}\right)^2} \times 100$$

$$= \sqrt{1.56 - 0.1936} \times 100$$

$= 1.1689 \times 100$

$= 116.89$

Hence, standard deviation $(\sigma) = 116.89$

Example 5

The salaries of employees in two industries are given in the table below. Determine which industry has a more uniform salary distribution:

Industry	Average monthly salary	Standard deviation of salary
A	Rs. 50,000	Rs. 100
B	Rs. 40,000	Rs. 110

Solution

Here, The average salary of industry A $(\overline{X}_1)$ = Rs. 50000

The average salary of industry B $(\overline{X}_2)$= Rs. 40000

The standard deviation of salaries in industry A(σ_1) = Rs. 100

The standard deviation of salaries in industry B = (σ_2) = Rs. 110

C.V. of industry A $= \frac{\sigma_1}{\overline{X}_1} \times 100\ \%$

$= \frac{100}{50000} \times 100\ \% = 0.2\ \%$

C.V. of industry B = (C.V.) $= \frac{\sigma_2}{\overline{X}_2} \times 100\ \%$

$= \frac{110}{40000} \times 100\ \% = 0.275\ \%$

Conclusion: Since the coefficient of variation of industry A is less than that of industry B, the salary distribution of employees in industry A is more uniform than in industry B.

Exercise 11.3

1. Tick (✓) the correct option for the given questions:

A. Which of the following is the standard deviation?

a. A measure of central tendency b. Mean

c. A measure of dispersion d. Median

B. What is the formula for calculating the coefficient of variation?

a. $\frac{\overline{X}}{\sigma}$ b. $\frac{\sigma}{\overline{X}}$ c. $\frac{\sigma}{\overline{X}} \times 100$ d. $\frac{\Sigma f|D|}{N}$

C. If the variance of a dataset (σ^2) = 64, what is the standard deviation?

a. 8 b. 16 c. 32 d. 4096

D. For two datasets, if the coefficient of variation (C.V.) is higher for one dataset than the other, how does it compare to the other dataset?

a. More uniform b. Less uniform

c. More dispersed d. No dispersion

E. If $\Sigma f(m - \overline{X})^2 = 6400$ and $N = 100$, what is the standard deviation?

a. 10 b. 64 c. 32 d. 8

2. a. What is standard deviation? Write.

b. What is the coefficient of variation? Write.

c. What is the difference between mean deviation and standard deviation? Wrtie.

d. What is the difference between the coefficient of standard deviation and the coefficient of variation? Write.

3. a. If $N = 100$, $\Sigma fm = 400$ and $\Sigma fm^2 = 4000$, calculate the mean and standard deviation.

b. If $N = 40$, $\Sigma fd = -400$ and $\Sigma fd^2 = 5000$, where $d = m - A$ and the assumed mean is A, calculate the standard deviation.

4. a. **For the given data, calculate the standard deviation and its coefficient.**

Weight (kg)	10 - 20	20 - 30	30 - 40	40 - 50	50 - 60
Number of students	5	10	12	7	6

b.

Obtained marks	10 - 20	20 - 30	30 - 40	40 - 50	50 - 60	60 - 70
Number of students	8	12	15	20	12	8

c.

Weight (kg)	3 - 5	5 - 7	7 - 9	9 - 11	11 - 13
Number of dogs	1	3	5	4	2

5. The marks obtained by 40 students in Mathematics examination of 50 full marks in a school are given.

Obtained marks	Less than 10	Less than 20	Less than 30	Less than 40	Less than 50
Number of students	5	11	25	35	40

a. Find the mean score.

b. Find the standard deviation and its coefficient.

6. In a factory, the daily wages of workers range from Rs. 600 to more than Rs. 1200. The data is given below:

Daily Wages (Rs.)	Number of Workers
More than 600	100
More than 700	80
More than 800	55
More than 900	25
More than 1000	15
More than 1100	8
More than 1200	4

a. Find the average wage.

b. Find the standard deviation and its coefficient.

c. Find the variance.

d. Find the coefficient of variation.

7. The monthly salaries of workers in two shoe factories are analyzed and presented in the following table:

	Factory A	Factory B
Number of workers	40	50
Average monthly salary (Rs.)	Rs. 35000	Rs. 45000
Standard deviation (Rs.)	Rs. 40	Rs. 35

a. Determine which factory spends more on salaries.

b. Determine which factory has a more uniform salary distribution and explain the reason.

8. In a school, 36 students of grade 10 appeared in a mathematics exam. Their scores are as follows:

35	62	72	50	70	74	34	60	36
21	14	75	48	28	38	64	21	45
57	40	19	34	30	47	20	75	51
45	56	22	29	22	13	53	66	62

a. Present the above data in a frequency table with the first-class interval as 10 - 20.

b. Find the average (mean) score.

c. Find the standard deviation and its coefficient.

d. Find the coefficient of variation.

Answer

1. A. c B. b C. b D. b E. d

2. Show to the teacher. 3. a. 4, 4.899 b. 11.53, 0.21

4. a. 12.35, 0.355 b. 14.74, 0.365 c. 2.22, 0.264

5. a. 26 b. 11.79, 0.453 6. a. 837 b. 155.34, 0.186 c. 24130.5156 d. 18.6%

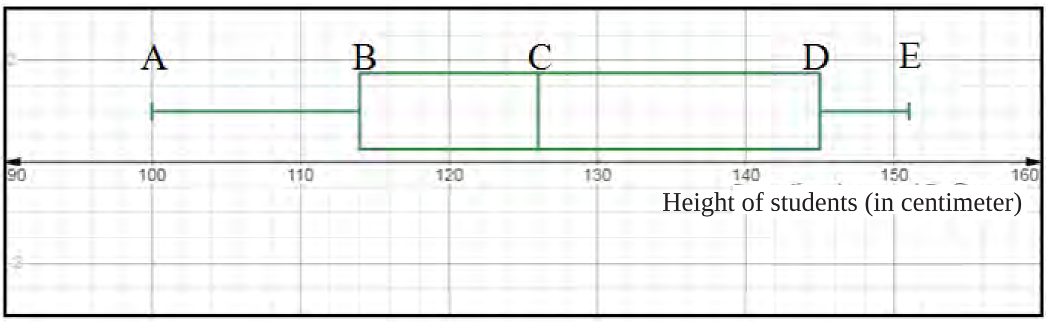
7. a. B b. B 8. a. Show to the teacher. b. 45.28 c. 19.07, 0.42 d. 42.1%

11.6 Whisker Box Plot

In statistics, a whisker box plot (box-and-whisker plot) is a reliable method used to understand the actual structure of data and to make meaningful comparisons between two groups. Sometimes, the mean (average) alone may hide the real differences within the data. However, a box plot provides a clear picture of the distribution and spread of the data.

Activity 1

We have already studied the whisker box plot in Grade 9. Based on that knowledge, observe the given graph and discuss the following questions to draw conclusions.



a. What type of graph does the above figure represent? Write.

b. In the above graph, what do the points A, B, C, D, and E represent?

c. What is the portion between the points A to B and D to E called?

d. What is the portion between the points B to D called?

e. What are the values of the points A, B, C, D and E in the figure?

The above figure is a whisker box plot. In any dataset, the middle 50% of the data is represented inside the box, and the remaining 50% of the data is represented by the lines extending on both sides. Such a graphical representation is called a whisker box plot.

The five points of a whisker box plot represent five important values of the dataset. In the figure, point A represents the smallest value, point B represents the first quartile (Q_1), point C represents the median (Q_2), point D represents the third quartile (Q_3), and point E represents the largest value.

In the given whisker box plot, the portions from A to B and from D to E are called the whiskers.

In the above whisker box plot, the heights of students (in cm) are represented. Therefore, in the given data: Minimum value = 100 cm, First quartile (Q_1) = 114 cm, Median (Q_2) = 126 cm, Third quartile (Q_3) = 145 cm, Maximum value = 151 cm

A whisker box plot is a graphical representation in which the middle 50% of the data is shown in a box, and the remaining 50% is represented by lines on both sides.

The five points of the whisker box plot represent the five values of the dataset, namely: minimum value, first quartile, median, third quartile, and maximum value.

Steps to Construct a Whisker Box Plot

Step 1: Arrange the given data in ascending order.

Step 2: Find the minimum value, median, quartiles, and maximum value of the data.

Step 3: Draw a scale on the graph and mark the five key values: minimum value, first quartile, median, third quartile, and maximum value on the graph.

Step 4: Draw a box between the first quartile (Q_1) and the third quartile (Q_3). Then, draw lines (whiskers) from the minimum value to Q_1 and from Q_3 to the maximum value.

Example 1

The ages (in years) of 11 students are given in the table below. From the given data, find the smallest value, first quartile, median, third quartile, and largest value and represent them in a whisker box plot.

13 17 16 14 11 13 10 16 11 18 12

Solution

Arranging the given data in ascending order:

10 11 11 12 13 13 14 16 16 17 18

Here, smallest value = 10

Now,

First quartile $(Q_1) = \left(\frac{N+1}{4}\right)^{th}$ item $= \left(\frac{11+1}{4}\right)^{th}$ item $= \left(\frac{12}{4}\right)^{th}$ item $= 3^{th}$ item

$\therefore Q_1 = 11$

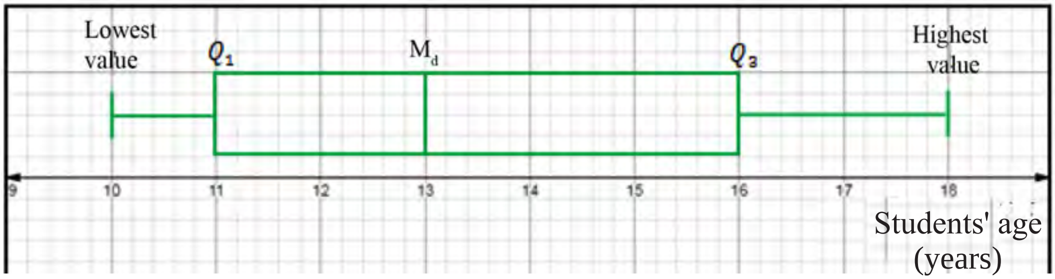
Median $(M_d) = \left(\frac{2(N+1)}{4}\right)^{th}$ item $= \left(\frac{2\times12}{4}\right)^{th}$ item $= 6^{th}$ item

$\therefore M_d = 13$

Third quartile $(Q_3) = \left(\frac{3(N+1)}{4}\right)^{th}$ item $= \left(\frac{3\times12}{4}\right)^{th}$ item $= 9^{th}$ item

$\therefore Q_3 = 16$ and largest value = 18

Now, using the smallest value, Q_1, median (M_d), Q_3 and the largest value obtained above, represent the given data in a whisker box plot.



11.2.2 Comparision of Two Data using Whisker Box Plot

Activity 2

The runs scored by two cricket players in the last 11 cricket matches are shown in the table below:

Player A	21	25	71	5	42	15	85	64	110	32	35
Palyer B	12	55	101	34	60	41	72	6	22	7	18

a. Find the smallest value, first quartile, median, third quartile, and largest value of the given two sets of data.

b. Represent the separate whisker box plots of the two data sets in a diagram.

c. On the basis of the whisker box plot, compare the scores obtained by player A and player B using the median and interquartile range.

For player A:

Arranging the scores of player A in ascending order: 5, 15, 21, 25, 32, 35, 42, 64, 71, 85, 110

Here, smallest value = 5

Now, first quartile (Q_1) = $\left(\frac{N+1}{4}\right)^{th}$ item = $\left(\frac{11+1}{4}\right)^{th}$ item = $\left(\frac{12}{4}\right)^{th}$ item = 3^{th} item

$\therefore Q_1 = 21$

Median (M_d) = $\left(\frac{2(N+1)}{4}\right)^{th}$ item = $\left(\frac{2\times12}{4}\right)^{th}$ item = 6^{th} item

$\therefore M_d = 35$

Third quartile (Q_3) = $\left(\frac{3(N+1)}{4}\right)^{th}$ item = $\left(\frac{3\times12}{4}\right)^{th}$ item = 9^{th} item

$\therefore Q_3 = 71$ and Largest value = 110

For player B:

Arranging the scores of player B in ascending order: 6, 7, 12, 18, 22, 34, 41, 55, 60, 72, 101

Here, smallest value = 6

Now, first quartile $(Q_1) = \left(\frac{N+1}{4}\right)^{th}$ item $= \left(\frac{11+1}{4}\right)^{th}$ item $= \left(\frac{12}{4}\right)^{th}$ item $= 3^{th}$ item

$\therefore Q_1 = 12$

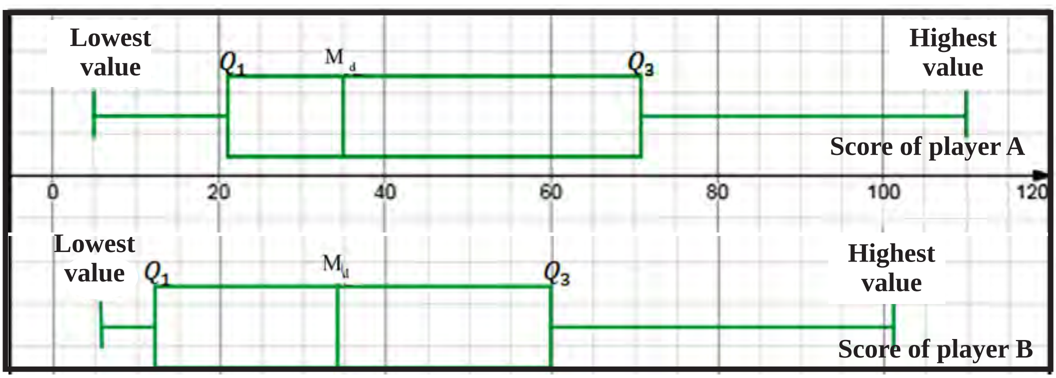
Median $(M_d) = \left(\frac{2(N+1)}{4}\right)^{th}$ item $= \left(\frac{2\times 12}{4}\right)^{th}$ item $= 6^{th}$ item

$\therefore M_d = 34$

Third quartile $(Q_3) = \left(\frac{3(N+1)}{4}\right)^{th}$ item $= \left(\frac{3\times 12}{4}\right)^{th}$ item $= 9^{th}$ item

$\therefore Q_3 = 60$ and largest value = 101

Now, using the smallest value, Q_1, M_d, Q_3, and the largest value of the scores of players A and B obtained above, represent the given data in the whisker box plot.



From the above two whisker box plots, the scores obtained by player A can be compared with the scores obtained by player B on the basis of the median and the interquartile range as follows.

a. On the basis of median

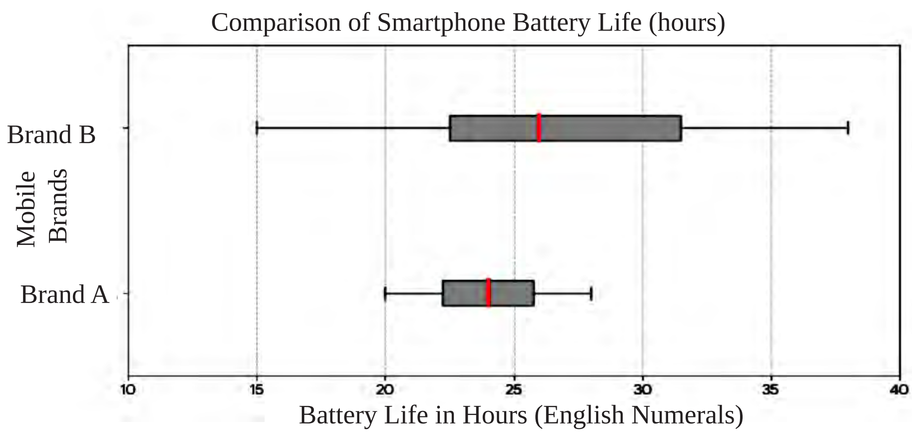
Here, the median of the scores obtained by player A is 35 and the median of the scores obtained by player B is 34. Therefore, the average score obtained by player A is greater than the average score obtained by player B.

b. On the basis of interquartile range

The interquartile range (IQR) of the scores obtained by player A = $Q_3 - Q_1 = 71 - 21 = 50$, and, the interquartile range (IQR) of the scores obtained by player B = $Q_3 - Q_1 = 60 - 12 = 48$. Since the interquartile range of the scores obtained by player A is greater than the interquartile range of the scores obtained by player B, the variation in player A's scores is greater than that of player B. That is, compared to player B, player A has less consistency in scores.

Exercise 11.2

1. The batteries of two mobile brands (brand A and brand B) were tested to find out how long they last after a full charge. Their box plots are given below:



A. In general, which brand's battery seems to last longer (on the basis of the median)?

a. Brand A　　b. Brand B　　c. Both are equal　　d. None

B. Which brand's battery shows greater consistency in battery life?

a. Brand B, because its range is large.

b. Brand A, because its box is smaller (interquartile range – IQR) and all the values are closer.

c. Both brands are equally reliable.

d. Brand B, because it provides battery backup up to 38 hours.

C. Which brand's battery has a higher guaranteed minimum battery life?

a. Brand B (15 hours)　　b. Brand A (20 hours)

c. Both have the same minimum battery life.

d. cannot say.

D. What does the red line in the box plot indicate?

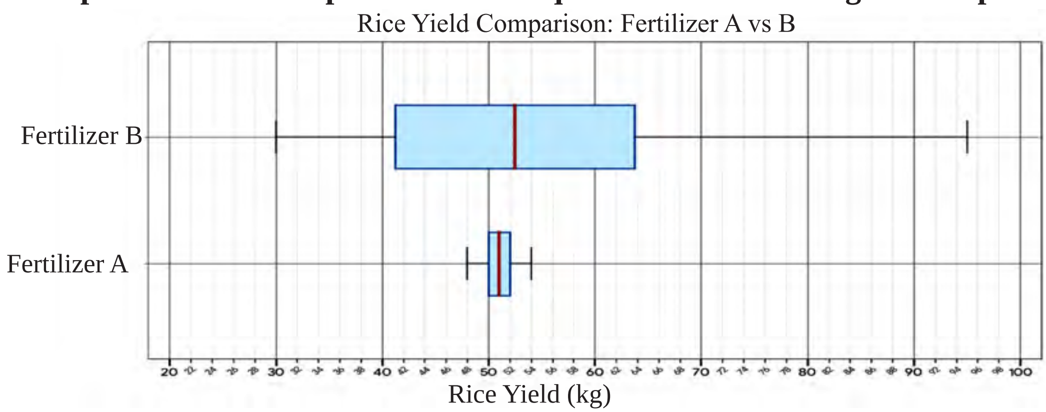
a. The maximum battery life of the battery

b. The minimum battery life of the battery

c. The median battery life (where 50% of the values lie)

d. The price of the battery.

2. The yield of paddy (in Kg) over ten years using two different fertilizers A and B is represented in a box plot. Answer the questions based on the given box plot.

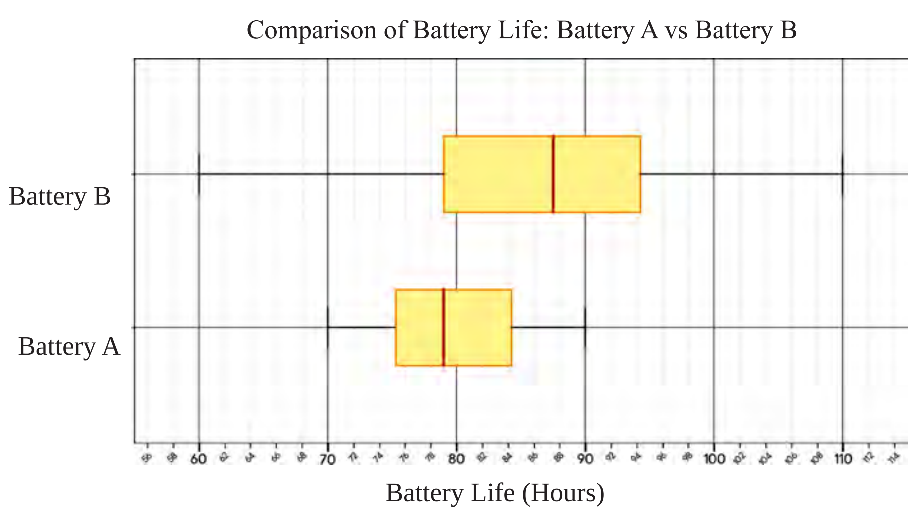


a. With which fertilizer is the minimum paddy yield higher?

b. If you are a farmer and want to minimize risk, which fertilizer would you choose, and why? (Hint: See the IQR.)

c. Is there any year in which fertilizer 'B' produced exceptionally high yield? How is this shown in the plot?

3. The given two box plots show the lifespan of two batteries. Answer the questions based on the box plots.



a. What is the median battery life of Battery A?

b. What is the median battery life of Battery B?

c. Find the range in battery life for both Battery A and Battery B.

d. Which battery would you choose to buy? Justify your decision with reasons.

4. The sales of mobiles in one week by two mobile shops are shown in the tables below:

Shop A:	8	5	14	9	11	7	18
Shop B:	10	12	8	16	6	3	2

a. From these two data sets, find the smallest value, first quartile, median, third quartile, and largest value.

b. Draw separate whisker box plots for the two data sets.

c. Based on the whisker box plots, compare the number of mobiles sold by shop A and shop B using the median and interquartile range.

Answer

1. A. b B. b C. b D. c 2 – 4. Show to the teacher.

Miscellaneous Exercise – Within Content Area

1. The marks obtained by students in an exam are as follows:

Obtained marks	20-30	30-40	40-50	50-60	60-70	70-80	80-90
Number of students	4	6	10	17	11	9	3

a. Write the relationship between the interquartile deviation, first quartile (Q_1), and third quartile (Q_3).

b. Based on the above data, find the standard deviation.

c. Find the coefficient of variation.

2. The marks obtained by 40 students of a class in the second term mathematics exam are given in the table below:

Obtained marks	20 - 30	30 - 40	40 - 50	50 - 60	60 - 70	70 - 80
Number of students	5	9	12	7	4	3

a. Write the formula to find the mean deviation from mean.

b. Find the standard deviation.

c. If the standard deviation of the first term exam marks was given as 8.75, compare the marks of the first and second term exams.

3. The monthly salary of 50 employees working in a bank are given in the table below:

Salary (Rs. in thousand)	More than 20,000	More than 30000	More than 40000	More than 50000	More than 60000	More than 70000
No. of employee	50	45	25	9	4	2

a. What is the quartile deviation? Write.

b. For the given data, which measure of dispersion is appropriate? Justify your answer.

c. From the data, find the quartile deviation and its coefficient.

4. The marks obtained by 40 students in a mathematics exam are as follows:

75	35	45	47	60	61	56	39	56	52
42	60	69	42	43	46	55	56	35	51
58	61	49	57	51	59	47	52	57	55
48	58	54	67	46	56	39	48	51	60

a. Write the formula to calculate the coefficient of variation.

b. Present the above data in a frequency table with class interval of 5.

c. Calculate the coefficient of variation.

d. When analyzing the exam results of two students, A and B, the coefficient of variation was found to be 21% and 12% respectively. Whose result shows more consistency? Justify your answer.

5. The marks obtained in Mathematics and Science by 11 students of grade 10 are given below:

Marks in Mathematics: 20, 93, 44, 58, 64, 68, 75, 82, 87, 23, 98

Marks in Science: 65, 19, 25, 31, 38, 45, 57, 35, 71, 15, 80

a. From the above two sets of data, find the minimum value, first quartile, median, third quartile, and maximum value.

b. Represent the information on the same graph paper using box plot.

c. Which subject has more consistency marks? Justify with reason.

Answer

1. Show to the teacher.

2. a. Show to the teacher. b. 14

c. The first term marks are reasonably consistent than second term marks.

3. a. Show to the teacher.

b. If open ended class, the interquartile deviation is appropriate. c. 7.03, 0.172

4. a and b. Show to the teacher. c. 16.86%

d. Student B's result shows greater consistency.

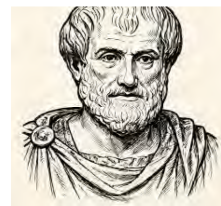
5. Show to the teacher.

Lesson 12

Continuity

12.1 Introduction

If a curve does not break at any point within a given interval, then, that curve is called a continuous curve. The ancient Greek mathematician Aristotle first discussed the idea of continuity. In the seventeenth century, Isaac Newton proved that motion is continuous. Later in the eighteenth century, Leonhard Euler explained how mathematical relations can be expressed continuously. After that, in the nineteenth century, Augustin-Louis Cauchy and Karl Weierstrass further developed the concept of continuity and established it as a fundamental idea of modern mathematics. Continuity is widely used in daily life as well as in mathematics, science and technology, engineering, business, economics, information and communication, and many other fields.



12.2 Continuity and Discontinuity

Activity 1

Observe the following figures and discuss the given questions.

Figure (a)

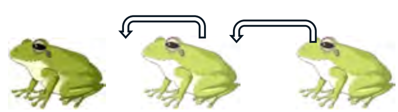


Figure (b)

a. What difference do you observe in the movement of a snake and a frog?

b. While moving, how do they relate with the ground?

c. Which animal's movement is continuous? Give reason.

When a snake moves, it touches all the points of the ground along its path and moves forward. But, when a frog jumps forward, it does not touch all the points of the ground along its path. Therefore, the path made by the snake shows continuity, whereas the path made by the frog shows discontinuity.

Activity 2

Try to draw the following figures without lifting your pen until the figure is completed.

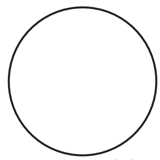


Figure (a)

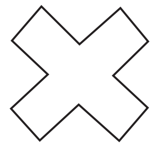


Figure (b)

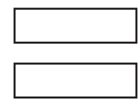


Figure (c)

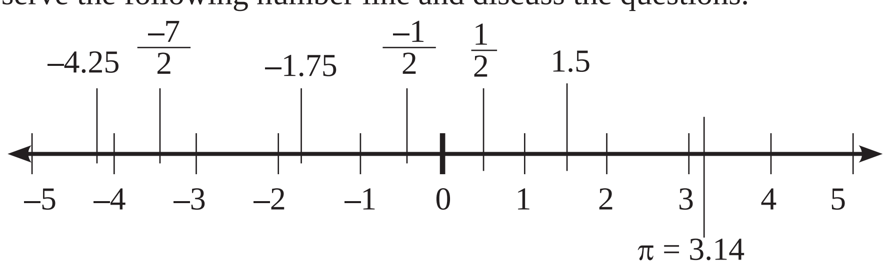
Which figures can you draw without lifting your pen?

Mathematicians use different terms to describe whether a pen must be lifted or not. In figures (a) and (b), the pen does not leave the paper from the beginning to the end, whereas in figure (c) the pen must be lifted at some point and placed again somewhere else. Therefore, figures (a) and (b) represent continuous curves, while figure (c) represents a discontinuous curve.

> If a figure can be drawn without lifting the pen from the beginning to the end, then the figure is continuous. If the pen must be lifted at some point and placed somewhere else, the figure is discontinuous.

Activity 3

Observe the following number line and discuss the questions.



a. Is there any natural number between 1 and 2 ? Do natural numbers form a continuous line ?

b. Is there any whole number between 3 and 4 ? Do whole numbers form a continuous line ?

c. Is there any integer between –1 and 0 ? Do integers form a continuous line ?

While moving from 1 to 2, or from 3 to 4, or from –1 to 0, there is an empty gap between them. Therefore, natural numbers, whole numbers, and integers do not form a continuous line.

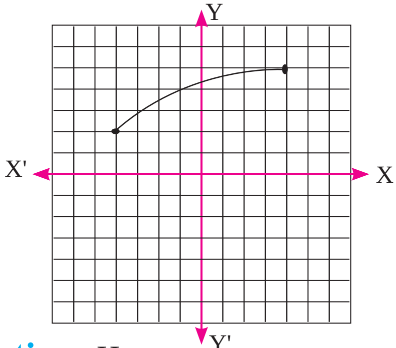
But when real numbers are placed on the number line, there is no empty space. Every point on the number line represents a real number. There is no gap or break. Therefore, the real number line is continuous.

Natural numbers, whole numbers, and integers do not form a continuous line, but real numbers show continuity on the number line.

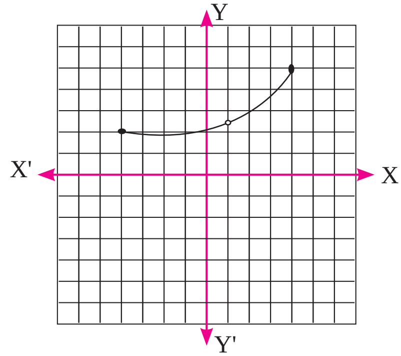
Example 1

From the given graphs, identify the intervals where the graph is continuous and the point where it is discontinuous.

a.



b.



Solution: Here,

a. The given graph continues without any break from $x = -4$ to 4.
Therefore, the graph is continuous in the interval: $-4 \leq x \leq 4$.

b. In the given graph, there is a break at $x = 1$ in the graph $x = -4$ to 4, but it is continuous at other points.

Therefore, the graph is discontinuous at $x = 1$ but continuous in the intervals $-4 \leq x < 1$ and $1 < x \leq 4$.

Exercise 12.1

1. Tick (✓) the correct option for the given questions.

A. If a curve can be drawn without lifting the pen, what is it called?

a. Discontinuous curve b. Continuous curve
c. Broken curve d. Cut curve

B. Which of the following may represent continuity in daily life?

a. Increase in the height of plants b. Increase in cricket score
c. Number of students in a class d. Number of coins in a pocket

C. Which parts of the number line are represented by the real numbers from –2 to +2?

a. Some points between –2 and +2 b. All points from –2 to +2
c. Only the points –2, 0 and +2 d. Only the points –2, –1, 0, +1, +2

D. For a function to be continuous at $x = a$, which of the following conditions is necessary?

a. The graph must be cut at $x = a$.

b. The graph must be broken at $x = a$.

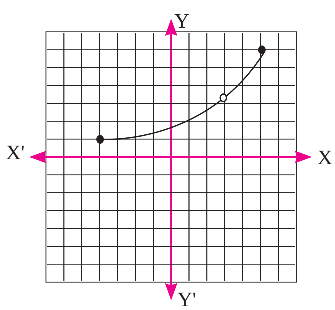
c. The graph must not be broken at $x = a$.

d. The graph must have a hole at $x = a$.

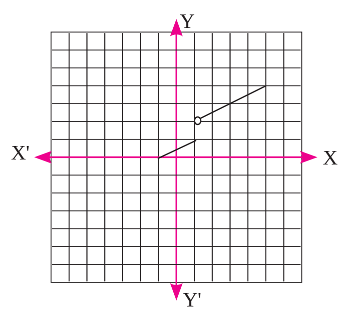
2. What is meant by a continuous curve and a discontinuous curve? Explain with examples.

3. What does it mean to say that a curve is continuous from –5 to +5? Explain with a diagram.

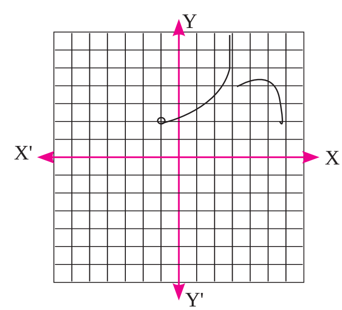
4. From the given graphs, write the intervals where the graphs are continuous and the points where they are discontinuous.



a.



b.



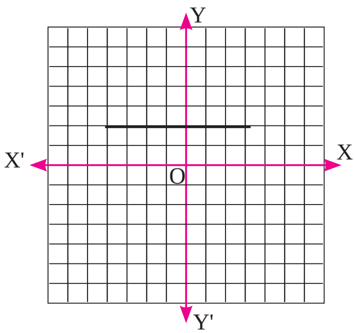
c.

5. The wiring system of a house is shown below:

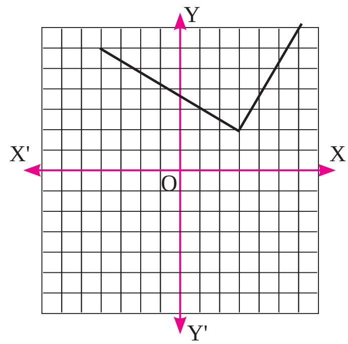
Wire A: This wire is straight and has no break, so electric current flows continuously.

Wire B: This wire is bent sharply at a 90° angle at a corner of the room, but the wire is not broken.

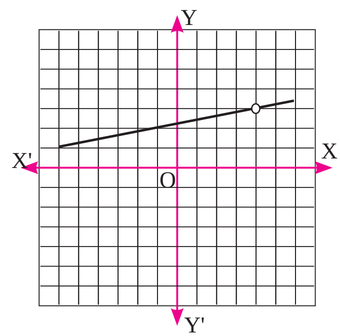
Wire C: This wire is broken exactly at a distance of 4 meters ($x = 4$), which stops the flow of electricity.



wire A



wire B



wire C

Answer the following:

a. Is wire A continuous or discontinuous? Give reason.

b. Is wire B continuous or discontinuous at $x = 3$? Give reason.

c. In mathematical language, what is the broken part of wire C at $x = 4$ called?

d. Compare wire B (sharp bend) and wire C (broken wire). In which case is the function (electric current) continuous and in which case discontinuous? Why does bending a wire not break continuity?

Answer

1. A. b B. a C. b D. c 2 - 5. Show to the teacher.

Project Work

Ask the age of people living in your neighborhood and fill the table below.

Age group (years)	0 - 20	20 - 40	40 - 60	60 - 80	above 80
Number of people	.....	.....	.....	.....	.....

Based on this data, draw the less-than and more-than cumulative frequency curves (ogives). Prepare a short report explaining where the curves are continuous or discontinuous, and present it in the classroom.

12.2 Continuity and Discontinuity of a Function

Activity 1

Observe the following graphs and discuss the given questions:

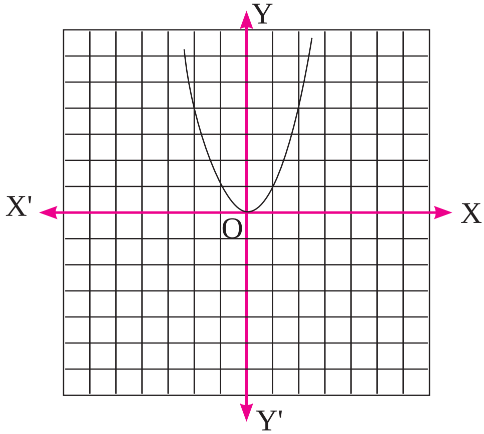


Figure No. 1 : $y = x^2$

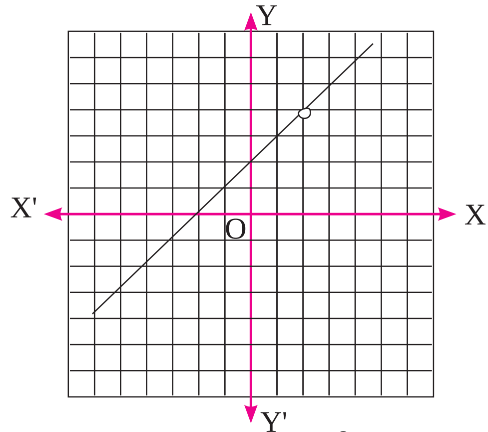


Figure No. 2 : $y = \frac{x^2 - 4}{x - 2}$

a. In the graph of figure 1, does the curve appear broken anywhere?

b. In the graph of figure 2, is the line broken at the position $x = 2$?

c. If $y = \frac{x^2 - 4}{x - 2}$ then, what is the value of y when $x = 2$? In such a situation, can the point (x, y) be shown in the graph? For the graph of $y = \frac{x^2 - 4}{x - 2}$ at the point $x = 2$, does the line appear connected or broken? Present your logic.

In the graph of figure 1, the curve does not appear broken anywhere. Here, the curve is continuous at every point. This means the curve is continuous throughout the entire interval.

In the graph of figure 2, the line is broken at the position $x = 2$. This means the curve has discontinuity at $x = 2$. At points other than $x = 2$, the curve is continuous.

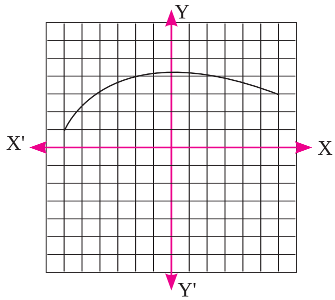
Activity 2

On a road, the condition of the road at three different places is as follows:

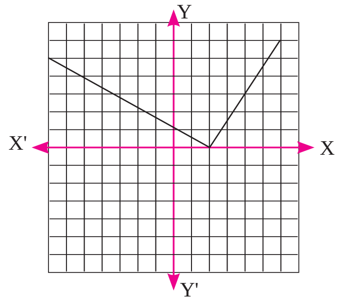
a. Condition A: The road is smooth and unbroken from beginning to end.

b. Condition B: The road has a sharp V-shaped bend, but the road is not broken anywhere.

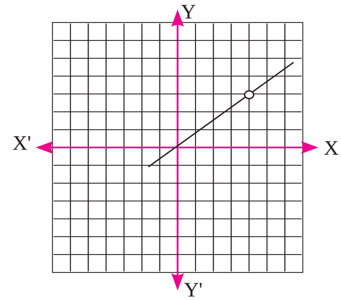
c. Condition C: Due to an earthquake, the road is broken exactly at 4 kilometers ($x = 4$), creating a gap there.



Condition A



Condition B



Condition C

Study the above three conditions and discuss the given questions.

a. What kind of graph is formed in condition A?

b. Compare condition B and condition C. Which one represents a continuous function and which represents a discontinuous function? Explain with reason.

c. A car is moving along road condition C. At $x = 4$ the road is broken. What type of sign would you make to indicate this break? Explain based on the given graphs.

d. In condition C, if the construction team fills the gap at $x = 4$ so that the road matches the same level as the road surface, analyze using the concept of $\lim_{x \to 4} f(x)$ and the value of the function $f(4)$.

In condition A, the road continues smoothly from beginning to end without any break. This is an example of a continuous function. In condition B, there is a sharp bend. This is also an example of a continuous function because even though the road has a sharp V-shaped turn, it is not broken anywhere. Thus, a vehicle can move forward without stopping. In condition C, there is a gap. This is an example of a discontinuous function, because at the point $x = 4$ the road is broken, forming a gap. From that point the vehicle cannot move forward directly. If the gap is filled, the road becomes continuous. From the graph of condition C, when x approaches 4, the height of the road $y = f(x)$ approaches 3. To solve this problem by making a table and finding the limit, observe the nature of the values of $f(x)$ as x approaches 4 from the right and left side.

Condition	Observing from the table values near $x = 4$:	Value of $f(x)$
Left side	3.9	2.9
	3.99	2.99
	3.999	2.999
In between	$x = 4$	....
Right side	4.001	3.001
	4.01	3.01
	4.1	3.1
	3.9	2.9

Left hand limit (LHL) : $\lim_{x \to 4^-} f(x) = 3$

Right hand limit (RHL) : $\lim_{x \to 4^+} f(x) = 3$

Conclusion: Since the road reaches the same height from both sides, the limit exists.

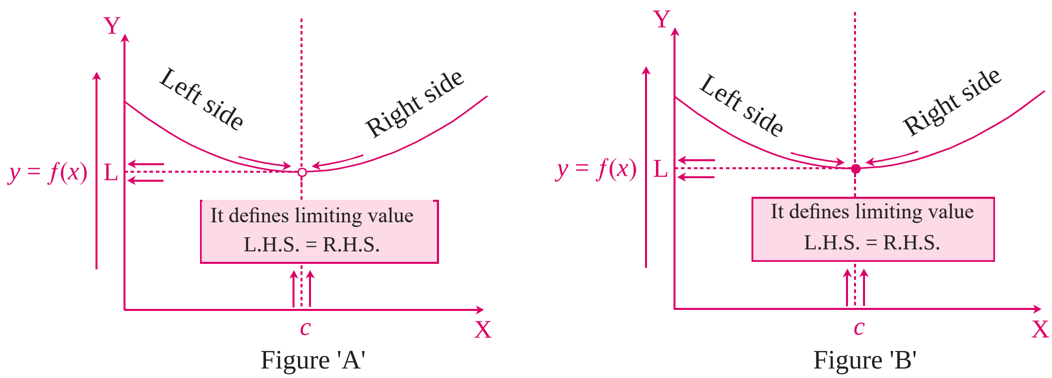
Thus, $\lim_{x \to 4} f(x) = 3$

If the construction team fills the gap exactly to the road surface level, then, the value of the function at $x = 4$ becomes $f(4) = 3$.

Hence, the road becomes continuous.

Activity 3

Observe the following two graphs and discuss the questions.



a. In both graphs (A and B), does the limit exist at the point $x = c$?

b. In both graphs, does the value of the function exist at the point $x = c$?

c. In which graph do both the function value and the limit exist and are equal at $x = c$?

d. In which graph is the function continuous at $x = c$?

From the above graphs, the limit exists at $x = c$ in both cases.

However, in graph (A), the value of the function does not exist at $x = c$ because the value of y at $x = c$ is not defined (shown by an empty circle).

In graph (B), the function value exists at $x = c$ because the value of y at $x = c$ is defined (shown by a filled circle).

Thus, in graph (B) the function is continuous at $x = c$.

Thought Provoking Question: Does the existence of a limit imply that the value of the function also exists?

From the above activities, we drew the following conclusion

For a function $f(x)$ to be continuous at $x = c$, the following three conditions must be satisfied:

1. The limit must exist $\lim_{x \to c} f(x)$.
2. The function value $f(c)$ must be defined. $x = c$.
3. The limit and functional value must be equal: $\lim_{x \to c} f(x) = f(c)$.

If any one of these three conditions is not satisfied, the function becomes discontinuous.

Example 1

Given a function: $f(x) = x + 1$

a. Find the values of $f(x)$ when $x = 1.9, 1.99, 1.999$ and 1.9999.

b. Find the values of $f(x)$ when $x = 2.1, 2.01, 2.001$,] and 2.0001.

c. Find $f(2)$.

d. Find the value of $\lim_{x \to 2^-} f(x)$ and $\lim_{x \to 2^+} f(x)$.

e. Find whether the function is continuous at $x = 2$.

Solution

a. If $f(x) = x + 1$, putting the value of $x = 1.9, 1.99, 1.999$ and 1.9999,

Then,

$f(1.9) = 1.9 + 1 = 2.9$

$f(1.99) = 1.99 + 1 = 2.99$

$f(1.999) = 1.999 + 1 = 2.999$

$f(1.9999) = 1.9999 + 1 = 2.9999$

b. Here, in $f(x) = x + 1$, putting the value of $x = 2.1, 2.01, 2.001$ and 2.0001 respectively.

$f(2.1)$	$= 2.1 + 1$	$= 3.1$
$f(2.01)$	$= 2.01 + 1$	$= 3.01$
$f(2.001)$	$= 2.001 + 1$	$= 3.001$
$f(2.0001)$	$= 2.0001 + 1$	$= 3.0001$

c. Here, $f(2) = 2 + 1 = 3$

d. Here, from (a), $\lim\limits_{x \to 2^-} f(x) = 3$ and from (b) $\lim\limits_{x \to 2^+} f(x) = 3$

e. Here, $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^+} f(x) = f(2)$. So $f(x)$ is continuous at $x = 2$.

Example 2

A function $f(x) = \begin{Bmatrix} x + 2, & 1 \le x \le 2 \\ 4x - 2, & x \ge 2 \end{Bmatrix}$ is given .

a. Find the value of $f(x)$ when $x = 1.99$.

b. Find the value of $f(x)$ when $x = 2.01$.

c. What are the values of $\lim\limits_{x \to 2^-} f(x)$ and $\lim\limits_{x \to 2^+} f(x)$? Find it. Does the function $f(x)$ continuous at $x = 2$? Give reason.

Solution

a. Here, when $x = 1.99$, then $f(x) = x + 2$.

Thus, $f(1.99) = 1.99 + 2 = 3.99$

b. Here, when $x = 2.01$, then $f(x) = 4x + 2$.

Thus, $f(2.01) = 4 \times 2.01 - 2 = 8.04 - 2 = 6.04$

c. Here, form (a) $\lim\limits_{x \to 2^-} f(x) = 4$ and from (b) $\lim\limits_{x \to 2^+} f(x) = 6$

so, $\lim\limits_{x \to 2^-} f(x) \neq \lim\limits_{x \to 2^+} f(x)$

Thus, $f(x)$ is not continuous at $x = 2$.

Example 3

Put the values of x in the function $f(x) = \dfrac{x^2 - 1}{x - 1}$ and test continuity at $x = 1$.

Solution

For any function $f(x)$ to be continuous at the point $x = 1$, it must satisfy:

$$f(1) = \lim_{x \to 1} f(x)$$

Taking the values of x very close to 1 (from the left and from the right) in

$$f(x) = \frac{x^2 - 1}{x - 1}$$

x (from left)	*f (x)*	*x (from right)*	*f (x)*
0.9	1.9	1.1	2.1
0.99	1.99	1.01	2.01
0.999	1.999	1.001	2.001
0.9999	1.9999	1.0001	2.0001

From the above table, we can clearly see that as the value of x gets closer and closer to 1 (from the left or from the right), the value of $f(x)$ approaches 2.

Therefore, $\lim_{x \to 1} f(x) = 2$

The value of the function at $x = 1$ is $(f(1)) = f(1) = \dfrac{1^2 - 1}{1 - 1} = \dfrac{0}{0}$.

This is an indeterminate form. Therefore, $f(x)$ is not defined at $x = 1$.

Here, $\lim_{x \to 1} f(x)$ exists and is equal to 2. However, the actual value of the function at $x = 1$, that is $f(1)$, is not defined.

Hence, $\lim_{x \to 1} f(x) \neq f(1)$

Therefore, the function is discontinuous at the point $x = 1$.

Example 4

Prove that the function $f(x) = \dfrac{x^2 + x - 6}{x + 3}$ is not continuous at point $x = -3$.

Solution

To find the limiting value of $\lim_{x \to -3} f(x)$ by using the values of x,

$$f(x) = \frac{x^2 + x - 6}{x + 3} = \frac{(x-2)(x+3)}{x+3} = x - 2 \quad (\text{if } x \neq -3)$$

Table: $x \to -3$ values of $f(x)$

x (from left)	$f(x) = x - 2$	x (from right)	$f(x) = x - 2$
– 3.1	– 5.1	– 2.9	– 4.9
– 3.01	– 5.01	– 2.99	– 4.99
– 3.001	– 5.001	– 2.999	–4.999
– 3.0001	– 5.0001	– 2.9999	–4.9999

From the table. $x \to -3$, then $f(x) \to -5$. Hence, $\lim_{x \to -3} f(x) = -5$

The functional value at $x = -3$ is $f(-3)$

Put $x = -3$ in the given function $f(x) = \dfrac{x^2 + x - 6}{x + 3}$

$$f(-3) = \frac{(3)^2 + (-3) - 6}{-3 + 3}$$

$$f(-3) = \frac{9 - 3 - 6}{0} = \frac{0}{0}$$

Here, $\frac{0}{0}$ is an indeterminate form. So, it is not defined at $x = -3$.

Here, $\lim_{x \to -3} f(x) = -5$ but function $f(-3)$ is not defined at $x = -3$.

Thus. $f(x) = \dfrac{x^2 + x - 6}{x + 3}$ is not continuous at $x = -3$. Proved.

Exercise 12.2

1. Tick (✓) the correct option for the given questions:

A. Which limit of the function $f(x)$ is represented by $\lim_{x \to a^-} f(x)$?

a. Right-hand limit b. Left-hand limit

c. Value of the function d. None of the above

B. Which condition must be satisfied for the limit of a function $f(x)$ to exist at a point $x = c$?

a. $\lim_{x \to c^-} f(x) > \lim_{x \to c^+} f(x)$ b. $\lim_{x \to c^-} f(x) < \lim_{x \to c^+} f(x)$

c. $\operatorname{im}_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)\,)$ d. $f(c) = 0$

C. What is the value of the function $f(x) = 3x$ at $x = 1$?

a. 1 b. 3 c. 0 d. 4

D. Which of the following conditions is necessary for a function $f(x)$ to be continuous at the point $x = a$?

a. $\lim_{x \to c^-} f(x) > \lim_{x \to c^+} f(x)$ b. $\lim_{x \to c^-} f(x) < \lim_{x \to c^+} f(x)$

c. $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(a)$ d. $f(c) = 0$

E. If $f(x) = x + 1$, then, what is the value of $\lim_{x \to 1} \frac{1}{f(x)}$?

a. 0 b. 1 c. $\frac{1}{3}$ d. $\frac{1}{2}$

2. What do $\lim_{x \to c^-} f(x)$, and $\lim_{x \to c^+} f(x)$ represent? Explain with examples.

3. Write the three necessary conditions for a function $f(x)$ to be continuous at the point $x = a$.

4. If the function $g(x) = x + 3$, then,

a. Find the values of $g(x)$ when $x = 2.9, 2.99, 2.999$.

b. Find the values of $g(x)$ when $x = 3.1, 3.01, 3.001$.

c. Find the value of $g(3)$.

d. Find the values of $\lim_{x \to 3^+} g(x)$ and $\lim_{x \to 3^-} g(x)$

e. Is $g(x)$ continuous at $x = 3$? Give reason.

5. If the function $h(x) = 2x - 1$, then,

a. Find the values of $h(x)$ when $x = 0.9, 0.99, 0.999$.

b. Find the values of $h(x)$ when $x = 1.1, 1.01, 1.001$.

c. Find the value of $h(1)$.

d. Find the values of $\lim_{x \to 1^+} h(x)$ and $\lim_{x \to 1^-} h(x)$.

e. Is $h(x)$ continuous at $x = 1$? Give reason.

6. A function $k(x)$ is defined as follows:

$$k(x) = \begin{cases} 3 - x, & 0 \le x \le 1 \\ x + 1, & x > 1 \end{cases}$$

a. Find the value of $k(x)$ when $x = 0.99$.

b. Find the value of $k(x)$ when $x = 1.01$.

c. Find the values of $\lim_{x \to 1^-} k(x)$ and $\lim_{x \to 1^+} k(x)$.

d. Is the function $k(x)$ continuous at $x = 1$? Verify.

7. A function $p(x)$ is defined as:

$$p(x) = \begin{cases} 2x + 1, & 1 \le x \le 3 \\ x + 4, & x > 3 \end{cases}$$

a. Find the value of $p(x)$ when $x = 2.99$.

b. Find the value of $p(x)$ when $x = 3.01$.

c. Find the values of $\lim_{x \to 3^-} p(x)$ and $\lim_{x \to 3^+} p(x)$

d. Check whether the function $p(x)$ is continuous at $x = 3$.

8. Test the continuity or discontinuity of the function $f(x)$ by substituting different values of x:

a. $f(x) = 2x + 5$, at $x = 1$ (b) $f(x) = \frac{x^2 - 4}{x - 2}$; at $x = 2$

c. $f(x) = \frac{x^3 - 27}{x - 3}$; at $x = 3$

9. A function $f(x)$ is defined as:

a. $f(x) = \begin{cases} 7x - 2, & x < 3 \\ 5, & x = 3 \\ 6x + 1, & x > 3 \end{cases}$

Test the continuity of the function $f(x)$ at $x = 3$.

b. $g(x) = \begin{cases} x + 2, & 0 \le x \le 1 \\ 3, & x > 1 \end{cases}$

Test the continuity of the function $f(x)$ at $x = 1$.

10. The function $f(x)$ is defined as:

$$f(x) = \begin{cases} 2x - 3, & x < 2 \\ 2k - 3, & x = 2 \\ x + k, & x > 2 \end{cases}$$

Find the value of k so that the function $f(x)$ is continuous at $x = 2$.

Answer

1. A. b B. c C. b D. c E. d 2 - 10. Show to the teacher.

Project Work

Measure the temperature of your locality for 6 hours of one day at every half-hour interval and collect the data. Draw the line graph of the collected data. Based on the graph, prepare a report explaining continuity and discontinuity to your classmates and present it in the classroom.

Miscellaneous Exercise – Within Content Area

1. The function $g(x) = x + 3$ is given.

   a. What does it mean for a curve to be continuous or discontinuous at a particular point? Explain clearly with examples.

   b. What is meant by saying that the curve is continuous from –4 to +4? Write.

   c. Is the given function $g(x)$ continuous at $x = 3$? Give reason.

2. The function $f(x) = 4x + 1$ is defined.

   a. Calculate the values of f(1.999) and f(2.001).

   b. Find the values of $\lim_{x \to 2^-} f(x)$ and $\lim_{x \to 2^+} f(x)$.

   c. Calculate the value of $f(2)$.

   d. Is the function $f(x)$ continuous at $x = 2$? Give reason.

3. The function $p(x) = \begin{cases} x + 5, & x \le 3 \\ 4x - 4, & x > 3 \end{cases}$ is given.

   a. Under what condition is a function $f(x)$ continuous at the point $x = a$? Write in symbolic form.

   b. Find the value of $f(x)$ when $x = 2.9999$.

   c. Find the value of $f(x)$ when $x = 3.0001$.

   d. Is the function $f(x)$ continuous at $x = 3$? Give reason.

4. A function $h(x) = \begin{cases} kx + 3, & x \le 1 \\ 2x - 1, & x > 1 \end{cases}$ is given.

   a. What is meant by a continuous function?

   b. Find the values of $\lim_{x \to 1^+} h(x)$ and $\lim_{x \to 1^+} h(x)$

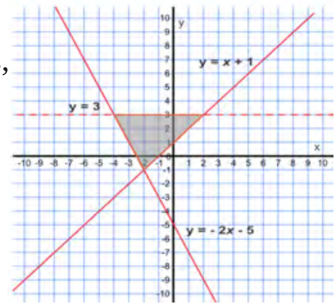
   c. If the function $h(x)$ is continuous at $x = 1$, find the value of k.

Answer

1–4. Show to your teacher.

Miscellaneous Exercise – Cross Content Area

1. In the given graph, triangle ABC represents the solution region of three inequalities.



   a. From the values satisfying all three inequalities, find the maximum and minimum values of $4x - 5y$.

   b. Find the angle between the lines BC and AC.

   c. If the function $y = f(x) = x + 1$ is defined, find the values of $f(2.999)$ and f(3.001).

   d. Is the function $f(x)$ continuous at $x = 3$? Give reason.

2. Given a function $f(x) = \begin{cases} 4x + 3 & x \leq 2 \\ 5x + 1 & x > 2 \end{cases}$

   a. If $f(x) = 5x + 1$, determine whether $f^{-1}(11)$ exists, and if it exists, find it.

   b. Test the continuity and discontinuity of $f(x)$ at $x = 2$.

3. For a continuous series, the linear relation between y and the variable is given by $5x -$ and the mean is 10.

   a. Find the value of y in terms of x.

   b. If $y = f(x)$, find the value of $f^{-1}(2)$.

   c. Find the equation of the line passing through the point (2, –3) and perpendicular to the line $5x - 1$.

4. If the function $g(x) = 5x + 2$ is defined,

   a. Find the values of $g(2.999)$ and $g(3.001)$.

   b. Is the function $g(x)$ continuous at $x = 3$? Give reason.

   c. If $g(x) = y$ represents a straight line, find the equation of the line parallel to it and passing through the point (2, –3).

   d. Find $g^{-1}(x)$.

5. The quadratic equation $3x^2 + 5x + 2 = 0$ is given.

   a. Solve the equation $3x^2 + 5x + 2 = 0$ graphically.

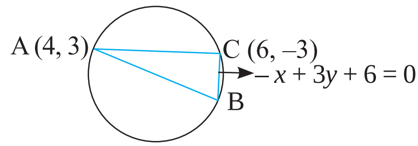
   b. The factors of $3x^2 + 5x + 2 = 0$ are $(3x + 2)$ and $(x + 1)$. Let them be $f(x)$ and $g(x)$ respectively. If $g \circ f^{-1}(x) = 2$, find the value of x.

   c. Find the equation of the line passing through the point (–3, –2) and perpendicular to the line $y = 3x + 2$.

6. In the given circle, AC, AB and BC are three chords.

Where A (4, 3) and C (6, –3) are given.

The equation of chord BC is $-x + 3y + 5 = 0$.



a. Find the equation of chord AC.

b. Find the angle between chords AC and BC.

c. Prove that BC is the diameter of the circle.

d. Express the equations of chords AC and BC respectively in the forms $y = f(x) = \frac{x-5}{3}$ and $y = g(x) = -3x + 15$ Then, find $g \circ f(2)$.

7. The vertices of triangle ABC are A (–4, 6), B (–6, –10) and C (12, –8).

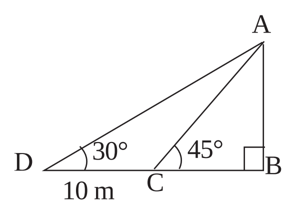
If R_1 represents reflection in the line $y = 0$ and R_2 represents reflection in the line $x - y = 0$.

a. Find the acute angle between the two lines representing R_1 and R_2.

b. Find the image of triangle ABC using the composite transformation $R_2 \circ R_1$.

c. Find the scalar product $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

8. In the given figure, AB is the height of a house. From points C and D on the same horizontal ground, the angles of elevation of the top of the house are 45° and 30° respectively and CD = 10 m.



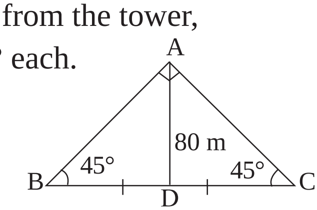
a. Write the definition of angle of elevation.

b. What is the length of AB? Find it.

c. Find the scalar product of $\overrightarrow{DA}$ and $\overrightarrow{DB}$.

d. What is the geometrical meaning of $\overrightarrow{DA} . \overrightarrow{DB}$? Write it.

9. In the given figure, AD is a tower of height 80 m. From two points B and C on the same horizontal ground in opposite directions from the tower, the angles of elevation of the top of the tower are 45° each.

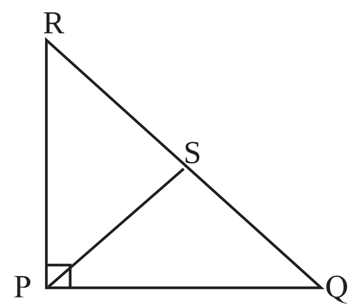


a. Write the definition of angle of depression.

b. Find the distance between the two men.

c. Prove by vector method that the midpoint D of the hypotenuse of a right triangle ABC is equidistant from all three vertices.

10. The coordinates of triangle ABC are A (2, 1), B (4, 2), C (1, 3).
    a. Find the matrix T representing a rotation of 90° about the origin.
    b. Find the transpose matrix T^t.
    c. Using matrix T^t, find the image coordinates of triangle ABC after transformation.
    d. Is the inverse matrix of T defined? Give reason.
11. The coordinates of triangle PQR are P (5, 1), Q (–4, 1) and R (5, 6). Where, $\angle P = 90°$ and the midpoint of QR is S.



    a. If the slopes of PQ and PR are m_1 and m_2 respectively, write the relation between m_1 and m_2.
    b. Find the scalar product of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$
    c. Find the equation of line QR.
    d. Prove by vector method that point S is equidistant from the vertices of triangle PQR.
12. The inequalities $3x + 2y + 4 \geq 0$, $3y \leq 6$, and $x \geq 0$ are given.
    a. Write the equations of the boundary lines.
    b. Represent the inequalities on the graph and find the solution region.
    c. Find any one angle of the polygon formed by the solution region.
    d. Using matrix $T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}$, find the coordinates of the vertices after transformation.
13. In triangle ABC, A (1, 3), B (5, 7), C (7, 3) are given. The midpoint of BC is D.
    a. If the position vectors of A and B are $\vec{a}$ and $\vec{b}$ what is the position vector of the midpoint of AB?
    b. Find the position vector of D.
    c. Using the scalar product, find the angle between vectors $\overrightarrow{AD}$ and $\overrightarrow{BC}$.
    d. Find the equation of AD.
14. The position $2\vec{i} + 3\vec{j}$, $6\vec{i} + 7\vec{j}$ and $8\vec{i} + 3\vec{j}$
    a. Find the equation of the median drawn from A to BC.
    b. Find the angle between the median drawn from A to BC and side BC.

15. The function $f(x) = (x + 2)^2 + 5$ is given.

a. Sketch the graph of the function.

b. State whether the graph is continuous or discontinuous. Give reason.

c. Are the values of $\lim_{x \to 3^-} f(x)$, $\lim_{x \to 3^+} f(x)$ and $f(3)$ equal? Give reason.

Answer

1 a. Max. value: –3 Min. value: – 31 b. 108.43° c. 3.999, 4.001 d. yes

2. a. Yes it is possible, $f^{-1}(11) = 2$ b. It is continuous.

3. a. $y = 50x - 10$ b. $f^1(x) = \frac{x + 10}{50}$ c. $x + 5y + 13 = 0$

4. a. 16.995, 17.005 b. $y \lim_{x \to 3^-} \lim_{x \to 3^+} f(x) = g(3) = 17$ c. $5x - y - 13 = 0$

(d) $g^{-1}(x) = \frac{x - 2}{5}$

5. a. $-1, -\frac{1}{2}$ b. 5 c. $x + 3y + 9 = 0$

6. a. $3x + y - 15 = 0$ b. 90° c. Show to the teacher. d. $gof(2) = 18$

7. a. 45° b. A' (–6, –4), B'(10, –6), C'(8, 12) c. 68

8. a. Show to the teacher. b. 13.66 *m* c. 559.80 d. Show to the teacher.

9. a. Show to the teacher. b. 160 *m* c. Show to the teacher.

10. a. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ b. $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ c. A'(1, –2), B'(2, –4) and C'(3, –1)

d. Inverse matrix of T is T^{-1} defined because $|T| \neq 0 =$

11. a. $m_1 \times m_2 = -1$ b. $\overrightarrow{PQ} . \overrightarrow{PR} = 0$ c. $5x - 9y + 29 = 0$
d. Show to the teacher.

12. a. $3x + 2y + 4 = 0$, $3y = 6$ and $x = 0$ b. Show to the teacher. c. 90^0

d. A'(8, 10), B'(–8, –10) and C'$\left(\frac{16}{3}, \frac{14}{3}\right)$

13. a. $\frac{(\vec{a} + \vec{b})}{2}$ b. $6\vec{i} + 5\vec{j}$ c. $\cos^{-1}\frac{1}{\sqrt{145}}$ d. $2x - 5y + 13 = 0$

14. a. $2x - 5y + 11 = 0$ b. $\cos^{-1}\frac{1}{\sqrt{145}}$

15. Show to the teacher.